—RAY'S ALGEBRA—
—JOSEPH RAY – M.D.—
— DATED 1848

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WITH

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DESIGNED FOR

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BY JOSEPH RAY, M. D.

PROFESSOR OF MATHEMATICS IN WOODWARD COLLEGE.

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PREFACE.

The object of the study of Mathematics, is two fold—the acquisition of useful knowledge, and the cultivation and discipline of the mental powers. A parent often inquires, "Why should my son study mathematics? I do not expect him to be a surveyor, an engineer, or an astronomer." Yet, the parent is very desirous that his son should be able to reason correctly, and to exercise, in all his relations in life, the energies of a cultivated and disciplined mind. This is, indeed, of more value than the mere attainment of any branch of knowledge.

The science of Algebra, properly taught, stands among the first of those studies essential to both the great objects of education. In a course of instruction properly arranged, it naturally follows Arithmetic, and should be taught immediately after it.

In the following work, the object has been, to furnish an elementary treatise, commencing with the first principles, and leading the pupil, by gradual and easy steps, to a knowledge of the elements of the science. The design has been, to present these in a brief, clear, and scientific manner, so that the pupil should not be taught merely to perform a certain routine of exercises mechanically, but to understand the *why* and the *wherefore* of every step. For this purpose, every rule is demonstrated, and every principle analyzed, in order that the mind of the pupil may be disciplined and strengthened so as to prepare him, either for pursuing the study of Mathematics intelligently, or more successfully attending to any pursuit in life.

Some teachers may object, that this work is too simple, and too easily understood. A leading object has been, to make the pupil feel, that he is not operating on unmeaning symbols, by means of arbitrary rules; that Algebra is both a rational and a practical subject, and that he can rely upon his reasoning, and the results

of his operations, with the same confidence as in arithmetic. For this purpose, he is furnished, at almost every step, with the means of testing the accuracy of the principles on which the rules are founded, and of the results which they produce.

Throughout the work, the aim has been, to combine the clear, explanatory methods of the French mathematicians, with the practical exercises of the English and German, so that the pupil should acquire both a practical and theoretical knowledge of the subject.

While every page is the result of the author's own reflection, and the experience of many years in the school-room, it is also proper to state, that a large number of the best treatises on the same subject, both English and French, have been carefully consulted, so that the present work might embrace the modern and most approved methods of treating the various subjects presented.

With these remarks, the work is submitted to the judgment of fellow laborers in the field of education.

Woodward College, August, 1848.

SUGGESTIONS TO TEACHERS.

It is intended that the pupil shall recite the Intellectual Exercises with the book open before him, as in mental Arithmetic. Advanced pupils may omit these exercises.

The following subjects may be omitted by the younger pupils, and passed over by those more advanced, until the book is reviewed.

Observations on Addition and Subtraction, Articles 60—64.

The greater part of Chapter II.

Supplement to Equations of the First Degree, Articles 164—177.

Properties of the Roots of an Equation of the Second Degree, Articles 215—217.

In reviewing the book, the pupil should demonstrate the rules on the blackboard.

The work will be found to contain a large number of examples for practice. Should any instructor deem these too numerous, a portion of them may be omitted.

To teach the subject successfully, the principles must be first clearly explained, and then the pupil exercised in the solution of appropriate examples, until they are rendered perfectly familiar.

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RAY'S

ALGEBRA.

PART FIRST.

INTELLECTUAL EXERCISES.

LESSON I

NOTE TO TEACHERS.—All the exercises in the following lessons can be solved in the same manner as in intellectual arithmetic; yet the instructor should require the pupils to perform them after the manner here indicated. In every question let the answer be verified.

1. I have 15 cents, which I wish to divide between William and Daniel, in such a manner, that Daniel shall have twice as many as William; what number must I give to each?

If I give William a *certain number*, and Daniel twice that number, both will have 3 times that *certain number;* but both together are to have 15 cents; hence, 3 times a certain number is 15.

Now, if 3 times a certain number is 15, one-third of 15, or 5, must be the number. Hence, William received 5 cents, and Daniel twice 5, or 10 cents.

If, instead of a *certain number*, we represent the number of cents William is to receive, by x, then the number Daniel is to receive will be represented by $2x$, and what both receive will be represented by x added to $2x$, or $3x$.

If $3x$ is equal to 15,
then $1x$ or x is equal to 5.

The learner will see that the two methods of solving this question are the same in principle; but that it is more convenient to represent the quantity we wish to find, by a single letter, than by one or more words.

In the same manner, let the learner continue to use the letter x to represent the smallest of the required numbers in the following questions.

NOTE.—x is read x, or one x, and is the same as $1x$. $2x$ is read two x, or 2 times x. $3x$ is read three x, or 3 times x, and so on.

2. What number added to itself will make 12?

Let x represent the number; then x added to x makes $2x$, which is equal to 12; hence if $2x$ is equal to 12, one x, which is the half of $2x$, is equal to the half of 12, which is 6.

VERIFICATION.—6 added to 6 makes 12.

3. What number added to itself will make 16?

If x represents the number, what will represent the number added to itself? What is $2x$ equal to? If $2x$ is equal to 16, what is x equal to?

4. What number added to itself will make 24?

5. Thomas and William each have the same number of apples, and they both together have 20; how many apples has each?

6. James is as old as John, and the sum of their ages is 22 years; what is the age of each?

7. Each of two men is to receive the same sum of money for a job of work, and they both together receive 30 dollars; what is the share of each?

8. Daniel had 18 cents; after spending a part of them, he found he had as many left as he had spent; how many cents had he spent?

9. A pole 30 feet high was broken by a blast of wind; the part broken off was equal to the part left standing; what was the length of each part?

Instead of saying *x added to x is equal to* 30, it is more convenient to say *x plus x* is equal to 30. To avoid writing the word *plus*, we use the sign $+$, which means the same, and is called the sign of *addition.* Also, instead of writing the word *equal*, we use the sign $=$, which means the same, and is called the sign of *equality.*

10. John, James, and Thomas, are each to have equal shares of 12 apples; if x represents John's share, what will represent the share of James? What will represent the share of Thomas? What expression will represent $x+x+x$ more briefly. If $3x=12$, what is the value of x? Why?

11. The sum of four equal numbers is equal to 20; if x represents one of the numbers, what will represent each of the others? What will represent $x+x+x+x$, more briefly? If $4x=20$, what is x equal to? Why?

12. What is $x+x$ equal to? Ans. $2x$.

13. What is $x+x+x$ equal to?

14. What is $x+x+x+x$ equal to?

LESSON II.

1. JAMES and John together have 18 cents, and John has twice as many as James; how many cents has each?

If x represents the number of cents James has, what will represent the number John has? What will represent the number they both have? If $3x$ is equal to 18, what is x equal to? Why?

NOTE.—If the pupil does not readily perceive how to solve a question, let the instructor ask questions similar to the preceding.

2. A travels a certain distance one day, and twice as far the next; in the two days he travels 36 miles; how far does he travel each day?

3. The sum of the ages of Sarah and Jane is 15 years, and the age of Jane is twice that of Sarah; what is the age of each?

4. The sum of two numbers is 16, and the larger is 3 times the smaller; what are the numbers?

5. What number added to 3 times itself will make 20?

6. James bought a lemon and an orange for 10 cents, the orange cost four times as much as the lemon; what was the price of each?

7. In a store-room containing 20 casks, the number of those that are full is four times the number of those that are empty; how many are there of each?

8. In a flock containing 28 sheep, there is one black sheep for each six white sheep; how many are there of each kind?

9. Two pieces of iron together weigh 28 pounds, and the heavier piece weighs three times as much as the lighter; what is the weight of each?

10. William and Thomas bought a foot-ball for 30 cents, and Thomas paid twice as much as William; what did each pay?

11. Divide 35 into two parts, such that one shall be four times the other.

12. The sum of the ages of a father and son is equal to 35 years, and the age of the father is six times that of his son; what is the age of each?

13. There are two numbers, the larger of which is equal to nine times the smaller, and their sum is 40; what are the numbers?

14. The sum of two numbers is 56, and the larger is equal to seven times the smaller; what are the numbers?

15. What is $x+2x$ equal to?

16. What is $x+3x$ equal to?

17. What is $x+4x$ equal to?

LESSON III.

1. Three boys are to share 24 apples between them; the second is to have twice as many as the first, and the third three times as many as the first. If x represents the share of the first, what will represent the share of the second? What will represent the share of the third? What is the sum of $x+2x+3x$? If $6x$ is equal to 24, what is the value of x? What is the share of the second? Of the third?

Verification.—The first received 4, the second twice as many, which is 8, and the third three times the first, or 12; and 4 added to 8 and 12, make 24, the whole number to be divided.

2. There are three numbers whose sum is 30, the second is equal to twice the first, and the third is equal to three times the first; what are the numbers?

3. There are three numbers whose sum is 21, the second is equal to twice the first, and the third is equal to twice the second. If x represents the first, what will represent the second? If $2x$ represents the second, what will represent the third? What is the sum of $x+2x+4x$? What are the numbers?

4. A man travels 63 miles in 3 days; he travels twice as far the second day as the first, and twice as far the third day as the second; how many miles does he travel each day?

5. John had 40 chestnuts, of which he gave to his brother a certain number, and to his sister twice as many as to his brother; after this he had as many left as he had given to his brother; how many chestnuts did he give to each?

6. A farmer bought a sheep, a cow, and a horse, for 60 dollars; the cow cost three times as much as the sheep, and the horse twice as much as the cow; what was the cost of each?

7. James had 30 cents; he lost a certain number; after this he gave away as many as he had lost, and then found that he had three times as many remaining as he had given away; how many did he lose?

8. The sum of three numbers is 36; the second is equal to twice the first, and the third is equal to three times the second; what are the numbers?

9. John, James, and William together have 50 cents; John has twice as many as James, and James has three times as many as William; how many cents has each?

10. What is the sum of x, $2x$, and three times $2x$?

11. What is the sum of twice $2x$, and three times $3x$?

LESSON IV.

1. If 1 lemon costs x cents, what will represent the cost of 2 lemons? Of 3? Of 4? Of 5? Of 6? Of 7?

2. If 1 lemon costs $2x$ cents, what will represent the cost of 2 lemons? Of 3? Of 4? Of 5? Of 6?

3. James bought a certain number of lemons at 2 cents a piece, and as many more at 3 cents a piece, all for 25 cents; if x represents the number of lemons at 2 cents, what will represent their cost? What will represent the cost of the lemons at 3 cents a piece? How many lemons at each price did he buy?

4. Mary bought lemons and oranges, of each an equal number; the lemons cost 2, and the oranges 3 cents a piece; the cost of the whole was 30 cents; how many were there of each?

5. Daniel bought an equal number of apples, lemons, and oranges for 42 cents; each apple cost 1 cent, each lemon 2 cents, and each orange 3 cents; how many of each did he buy?

6. Thomas bought a number of oranges for 30 cents, one-half of them at 2, and the other half at 3 cents each; how many oranges did he buy? Let $x=$ one-half the number.

7. Two men are 40 miles apart; if they travel toward each other at the rate of 4 miles an hour each, in how many hours will they meet?

8. Two men are 28 miles asunder; if they travel toward each other, the first at the rate of 3, and the second at the rate of 4 miles an hour, in how many hours will they meet?

9. Two men travel toward each other, at the same rate per hour, from two places whose distance apart is 48 miles, and they meet in six hours; how many miles per hour does each travel?

10. Two men travel toward each other, the first going twice as fast as the second, and they meet in 2 hours; the places are 18 miles apart; how many miles per hour does each travel?

11. James bought a certain number of lemons, and twice as many oranges, for 40 cents; the lemons cost 2, and the oranges 3 cents a piece; how many were there of each?

12. Two men travel in opposite directions; the first travels three times as many miles per hour as the second; at the end of 3 hours they are 36 miles apart; how many miles per hour does each travel?

13. A cistern, containing 100 gallons of water, has 2 pipes to empty it; the larger discharges four times as many gallons per

hour as the smaller, and they both empty it in 2 hours; how many gallons per hour does each discharge?

14. A grocer sold 1 pound of coffee and 2 pounds of tea for 108 cents, and the price of a pound of tea was four times that of a pound of coffee: what was the price of each?

If x represents the price of a pound of coffee, what will represent the price of a pound of tea? What will represent the cost of both the tea and coffee?

15. A grocer sold 1 pound of tea, 2 pounds of coffee, and 3 pounds of sugar, for 65 cents; the price of a pound of coffee was twice that of a pound of sugar, and the price of a pound of tea was three times that of a pound of coffee. Required the cost of each of the articles.

If x represents the price of a pound of sugar, what will represent the price of a pound of coffee? Of a pound of tea? What will represent the cost of the whole?

LESSON V.

1. James bought 2 apples and 3 peaches, for 16 cents; the price of a peach was twice that of an apple; what was the cost of each?

If x represents the cost of an apple, what will represent the cost of a peach? What will represent the cost of 2 apples? Of 3 peaches? Of both apples and peaches?

2. There are two numbers, the larger of which is equal to twice the smaller, and the sum of the larger and twice the smaller is equal to 28; what are the numbers?

3. Thomas bought 5 apples and 3 peaches for 22 cents; each peach cost twice as much as an apple; what was the cost of each?

4. William bought 2 oranges and 5 lemons for 27 cents; each orange cost twice as much as a lemon; what was the cost of each?

5. James bought an equal number of apples and peaches for 21 cents; the apples cost 1 cent, and the peaches 2 cents each; how many of each did he buy?

6. Thomas bought an equal number of peaches, lemons, and oranges, for 45 cents; the peaches cost 2, the lemons 3, and the oranges 4 cents a piece; how many of each did he buy?

7. Daniel bought twice as many apples as peaches for 24 cents; each apple cost 2 cents, and each peach 4 cents; how many of each did he buy?

8. A farmer bought a horse, a cow, and a calf, for 70 dollars; the cow cost three times as much as the calf, and the horse twice as much as the cow; what was the cost of each?

9. Susan bought an apple, a lemon, and an orange, for 16 cents; the lemon cost three times as much as the apple, and the orange as much as both the apple and the lemon; what was the cost of each?

10. Fanny bought an apple, a peach, and an orange, for 18 cents; the peach cost twice as much as the apple, and the orange twice as much as both the apple and the peach; what was the cost of each?

LESSON VI.

1. James bought a lemon and an orange; the orange cost twice as much as the lemon, and the difference of their prices was 2 cents; what was the cost of each?

If x represent the cost of the lemon, what will represent the cost of the orange? What is $2x$ less x represented by?

2. What is $3x$ less x represented by? What is $3x$ less $2x$ represented by?

What is $4x$ less x represented by? What is $5x$ less $2x$ represented by?

The word *minus*, is used instead of *less;* and the sign —, for the sake of brevity, is used to avoid writing the word minus.

Thus, if we wish to take the difference between $3x$ and x, we may say,

$3x$ less x,

or $3x$ *minus* x; which may be written $3x-x$.

When the sign — is used, it is to be read *minus*.

3. Thomas bought a lemon and an orange; the orange cost three times as much as the lemon, and the difference of their prices was 4 cents; what was the price of each? If x represents the cost of the lemon, what will represent the cost of the orange? What is $3x-x$ represented by?

4. In a school containing classes in Grammar, Geography, and Arithmetic, there are three times as many studying Geography as Grammar, and twice as many studying Arithmetic as Geography; there are 10 more in the class in Arithmetic than in that in Grammar; how many more are there in each class? If x represents the number in the class in Grammar, what will represent the number

in the class in Geography? In the class in Arithmetic? What is $6x—x$ represented by? What is it equal to?

5. The age of Sarah is three times the age of Jane, and the difference of their ages is 12 years; what is the age of each?

6. The difference of two numbers is 28, and the greater is equal to eight times the less; what are the numbers?

7. Daniel has four times as many cents as William, and Joseph has twice as many as both of them; but if twice the number of Daniel's cents be taken from Joseph's, the remainder is only 16; how many cents has each?

8. Susan bought a lemon, an orange, and a pine-apple; the orange cost twice as much as the lemon, and the pine-apple three times as much as both the lemon and the orange; the pine-apple cost 14 cents more than the orange; what was the cost of each?

9. James bought 1 lemon and 2 oranges; an orange cost twice as much as a lemon, and the difference between the cost of the oranges and the lemon was 6 cents; what was the cost of each?

10. Charles bought 2 lemons and 3 oranges; an orange cost twice as much as a lemon, and the difference between the cost of the lemons and the oranges was 8 cents; what was the cost of each?

11. A man bought a cow, a calf, and a horse; the cow cost twice as much as the calf, and the horse twice as much as the cow; the difference between the price of the horse and that of the calf was 30 dollars; what was the cost of each?

12. There are three numbers, of which the second is three times the first, and the third is twice as much as both the first and second, while the difference between the second and third is 10; what are the numbers?

LESSON VII.

1. James and John together have 11 cents, and John has 3 more than James; how many has each?

If James has x cents, then John has $x+3$, and they both have $x+x+3$, or $2x+3$ cents; hence, $2x+3$ are equal to 11; hence, if $2x$ and 3 are equal to 11, $2x$ must be equal to 11 less 3, which is equal to 8; then, if $2x$ is equal to 8, one x, or x, must be equal to 4.

2. William and Daniel together have 9 apples, and Daniel has one more than William; how many has each? If x represents

the apples William has, what will represent the apples Daniel has? What will represent the number they both have?

3. In a class containing 13 pupils, there are three more boys than girls; how many are there of each?

4. In a store-room containing 40 barrels, the number of those that are empty exceeds the number filled by 10; how many are there of each?

5. In a flock of fifty sheep, the number of those that are white exceeds the number that are black, by 30; how many are there of each kind?

6. Two men together can earn 60 dollars in a month, but one of them can earn 10 dollars more than the other; how many dollars can each earn?

7. The sum of two numbers is 25, and the larger exceeds the smaller by 15; what are the numbers?

8. Sarah and Jane bought a toy for 25 cents, of which Jane paid 5 cents more than Sarah; how much did each pay?

9. The difference between two numbers is 4, and their sum is 16; what are the numbers? If x represents the smaller number, what will represent the larger?

10. The difference between two numbers is 5, and their sum is 35; what are the numbers?

LESSON VIII.

1. James and John together have 15 cents, and John has twice as many as James, and 3 more; how many has each?

If x represents the number James has, then $2x+3$ will represent the number John has, and $x+2x+3$, or $3x+3$, what they both have. If $3x+3$ is equal to 15, then $3x$ must be equal to 15 *less* 3, or 12; hence x is equal to 4, the number James has; then John has 11.

2. William bought a lemon and an orange for 7 cents; the orange cost twice as much as the lemon and 1 cent more; what was the cost of each?

3. There are two numbers whose sum is 35; the second is twice the first and 5 more; what are the numbers?

4. In an orchard containing apple-trees and cherry-trees, the number of apple-trees is three times that of the cherry-trees, and 7 more; the whole number of trees in the orchard is 51; how many are there of each kind?

5. A farmer bought a cow and a calf, for 13 dollars; the cow cost three times as much as the calf, and 1 dollar more; what was the cost of each?

6. William and Thomas gave 50 cents to a poor woman; William gave twice as many as Thomas, and 5 cents more; how many cents did each give?

7. Eliza and Jane bought a doll for 14 cents; Eliza paid twice as much as Jane, and 2 cents more; what did each pay?

8. Divide the number 15 into two parts, so that one part shall exceed the other by 3.

9. Divide the number 26 into two parts, so that the greater part shall be 5 more than twice the less part.

10. The sum of two numbers is 23, and the greater is equal to three times the less, and 3 more; what are the numbers?

11. Two numbers added together make 40; the greater is 5 times the less, and 4 more; what are the numbers?

12. A man has two flocks of sheep; the larger contains six times as many as the smaller, and 5 more, and the number in both is 82; how many are there in each?

LESSON IX.

1. James has as many cents as John, and 2 more, and Thomas has as many as John, and 3 more; they all have 26 cents; how many has each? If x represents the number of cents John has, what will represent the number James has? The number Thomas has? The number they all have?

2. James, Thomas, and John, went out to gather chestnuts; Thomas gathered 5 more than James, and John 3 more than Thomas, and they all gathered 34; how many did each gather?

3. A father distributed 25 cents among his three boys; to the second he gave 2 more than to the first, and to the third, 3 more than to the second; how many did he give to each?

4. Divide the number 19 into three parts, so that the first may be 2 more than the second, and the third twice as much as the second, and 1 more.

5. Divide 13 apples between three boys, so that the second shall have 1 more than the first, and the third, 2 more than the second.

6. A peach, a lemon, and an orange, cost 15 cents; the lemon cost 1 cent more than twice as much as the peach, and the orange

2 cents more than three times as much as the peach; how many cents did each cost?

7. Three pieces of lead together weigh 47 pounds; the second is twice the weight of the first, and the third weighs 7 pounds more than the second; what is the weight of each piece?

8. The sum of the ages of Eliza, Jane, and Sarah, is 38 years; Jane is 3 years older than Eliza, and Sarah is 2 years older than Jane; what are their ages?

9. A father has three sons, each of whom is 2 years older than his next younger brother, and the sum of their ages is 27 years; what is the age of each?

10. The sum of three numbers is 29; the second is twice the first and 1 more, and the third is equal to the second, and 2 more; what are the numbers?

11. A man bought 2 pounds of coffee and 1 pound of tea, for 50 cents; the price of a pound of tea was 10 cents more than twice the price of a pound of coffee; what did each cost?

12. A man bought 3 pounds of coffee and 1 pound of tea, for 77 cents; the price of a pound of tea was equal to the price of 2 pounds of coffee, and 7 cents more; what was the price of each?

13. Says A to B, "Good morning, master, with your hundred geese." Says B, "I have not 100; but, if I had twice as many as I now have, and 20 more, I should have 100." How many had he?

LESSON X.

1. If $x+1$ represent a certain number, what will represent twice that number? Since twice x is $2x$, and twice 1 is 2, twice $x+1$, will be represented by $2x+2$.

2. What is 3 times $x+1$? 4 times $x+1$? 5 times $x+1$?

3. If $x+2$ represent a certain number, what will represent twice that number? 2 times x is $2x$, and 2 times 2 is 4, hence, twice $x+2$ is is $2x+4$.

4. What is 3 times $x+2$? 4 times $x+2$? 5 times $x+2$?

5. If $2x+1$ represent a certain number, what will represent twice that number? Twice $2x$ is $4x$, and twice 1 is 2, hence, twice $2x+1$ is $4x+2$.

6. What is 3 times $2x+1$? 4 times $2x+1$? 5 times $2x+1$?

7. What is 2 times $3x+2$? 3 times $3x+2$? 4 times $3x+2$?

8. What is x, $x+1$, and $x+2$ equal to?

9. What is x, $x+1$, and $3x+3$ equal to?

10. What is x, $x+3$, and $2x+2$ equal to?

11. A father divided 15 cents between his three boys; giving to the second 1 more than to the first, and to the third twice as many as to the second; how many cents did each receive?

12. The sum of 3 numbers is 34; the second is 1 more than the first, and the third is 3 times the second; what are the numbers?

13. Eliza, Jane, and Sarah, together have 24 cents; Jane has twice as many as Eliza, and 1 more, and Sarah has twice as many as Jane; how many cents has each?

14. A man bought 1 pound of coffee and 2 pounds of tea, for 62 cents; the price of a pound of tea was equal to that of 2 pounds of coffee, and 1 cent more; what was the cost of each?

15. A man worked three days for 10 dollars; the second day he earned 1 dollar more than the first, and the third day as much as both the first and second; how much did he earn each day?

16. Three boys together spent 43 cents; the second spent 5 cents more than the first, and the third twice as much as the second; how many cents did each spend?

17. Divide the number 33 into three parts, so that the second shall be 2 more than the first, and the third equal to five times the second.

18. Three men, A, B, and C, have 40 dollars between them; B has twice as many as A, and 1 dollar more, and C has 3 times as many as B; how many dollars has each?

19. Divide the number 29 into three parts, such that the second shall be equal to the first, and 1 more, and the third equal to three times the second.

20. A man bought 3 pounds of sugar and 2 pounds of coffee, for 41 cents; the price of a pound of coffee was 3 cents more than that of a pound of sugar; what was the cost of each?

21. James bought 2 lemons and 3 oranges, for 27 cents; an orange cost twice as much as a lemon, and 1 cent more; what was the cost of each?

22. An apple, a peach, and 2 pears, cost 17 cents; the peach cost 1 cent more than the apple, and each pear twice as much as the peach; what was the cost of each?

23. An apple, 2 peaches, and 3 pears, cost 14 cents; a peach cost 1 cent more than the apple, and a pear 1 cent more than a peach; what was the cost of each?

24. Two pears, 3 lemons, and 4 oranges, cost 29 cents; a lemon cost 1 cent more than a pear, and an orange 1 cent more than a lemon; what was the cost of each?

LESSON XI.

1. James has 4 cents, and John has 1 cent less than James; how many cents has John? What is 1 less than 4? What is 2 less than 4?

2. If x represents a certain number, what will represent 1 less than that number? Ans. $x-1$; read x minus 1.

3. If x represents a certain number, what will represent 2 less than that number? What will represent 3 less than that number?

4. If a certain number less 1 is equal to 3, what is the number equal to?

5. If $x-1$ is equal to 3, what is x equal to?

6. If $2x-1$ is equal to 5, what is $2x$ equal to? If $2x$ is equal to 6, what is x equal to?

7. If $3x-2$ is equal to 10, what is $3x$ equal to? If $3x$ is equal to 12, what is x equal to?

8. If $5x-3$ is equal to 17, what is $5x$ equal to? If $5x$ is equal to 20, what is x equal to?

9. James and John together have 17 cents, and James has 3 cents less than John; how many has each?

If x represents the number of cents John has, what will represent the number James has? What is x and $x-3$ equal to? If $2x-3$ is equal to 17, what $2x$ equal to? If $2x$ is equal to 20, what is x equal to?

10. Divide the number 17 into two parts, so that one shall be 5 less than the other.

11. An orange and a lemon together cost 8 cents, and the lemon cost two cents less than the orange; what was the cost of each?

12. The sum of two numbers is 20, and the smaller is 4 less than the greater; what are the numbers?

13. William and Daniel together have 20 cents, and Daniel has twice as many as William, wanting 1 cent; how many cents has each?

14. The sum of two numbers is 24, and the larger is twice the smaller, wanting 3; what are the numbers?

15. In a basket containing 25 apples and peaches, if 5 be subtracted from twice the number of apples, it will give the number of peaches; how many are there of each?

16. The sum of two numbers is 25, and the greater is equal to 3 times the smaller, wanting 7; what are the numbers?

17. A school contains 37 pupils, the number of boys is 3 times the number of girls, wanting 3; what is the number of each?

18. A cow, a calf, and a sheep, cost 28 dollars; the sheep cost 2 dollars less than the calf, and the cow cost 4 times as much as the calf; what was the cost of each?

LESSON XII.

1 What number is that, to which if 3 be added, the number will be doubled? If x represents the number, what will $x+3$ be equal to?

Since $2x$ is equal to $x+3$, it is plain that x is equal to 3.

2. What number is that, to which if 5 be added, the number will be doubled?

3. What number is that, to which if 4 be added, the sum will be 3 times the number? If x represents the number, $x+4$ will be equal to $3x$; but if $3x$ is equal to $x+4$, it is plain that $2x$ is equal to 4, and that x is equal to 2.

4. What number is that, to which if 9 be added, the sum will be 4 times the number? If x represents the number, what will $x+9$ be equal to? If $4x$ is equal $x+9$, it is plain that $3x$ is equal to 9, and that x is equal to 3.

5. What number is that, to which if 15 be added, the sum will be four times the number?

6. There are 10 years difference between the ages of two brothers, and the age of the elder is 3 times that of the younger; what is the age of each?

7. James says to John, "I have 4 times as many apples as you have; but if you had 9 apples more than you now have, we would then each have an equal number." How many has each?

8. The difference of two numbers is 20, and the greater is 5 times the smaller; what are the numbers?

9. The age of Eliza exceeds that of Jane 16 years, while the age of the former is five times that of the latter; what are their ages?

10. James bought a book and a toy; the book cost six times as much as the toy, and the difference of their prices was 20 cents; how much did he pay for each?

11. The difference between the age of a father and that of his son, is 30 years, and the age of the father is seven times the age of the son; what are their ages?

12. What number is that, to which if 32 be added, the sum will be equal to nine times the number itself?

13. What number is that which is 6 less than 3 times the number itself?

14. James is 12 years younger than John; but John is only four times the age of James; what are their ages?

15. What number is that, to the double of which, if 8 be added, the sum will be equal to 4 times the number?

In this case, if x represents the number, $4x$ is equal to $2x+8$; hence $2x$ must be equal to 8, and x equal to 4.

16. What is the value of x, when $5x$ is equal to $3x+6$?

17. What is the value of x, when $5x$ is equal to $2x+15$?

18. What is the value of x, when $8x$ is equal to $3x+15$?

19. What is the value of x, when $10x$ is equal to $4x+24$?

20. What number is that, to the double of which, if 21 be added, the sum will be five times the number?

21. If Daniel's age be multiplied by 4, and 30 added to the product, the sum will be 6 times his age; what is his age?

22. What number added to twice itself and 32 more, will make a sum equal to 7 times the number?

23. What number added to itself and 40 more, will make a sum equal to 10 times the number?

24. A father gave his son 3 times as many cents as he then had, his uncle then gave him 40 cents, when he found he had 9 times as many as at first; how many had he at first?

LESSON XIII.

1. What number is that which being increased by 5, and then doubled, the sum will be equal to three times the number?

In this example let x represent the number, then $x+5$ doubled, will be $2x+10$, which is equal to $3x$; hence x is equal to 10.

2. Sarah is 2 years older than Jane, and twice Sarah's age is equal to three times the age of Jane; what is the age of each?

3. William has 8 cents more than Daniel, and three times William's money is equal to 5 times that of Daniel; how many cents has each?

4. Three pounds of coffee cost as much as 5 pounds of sugar, and 1 pound of coffee cost 6 cents more than 1 pound of sugar; what is the price of a pound of each?

5. A farmer bought 2 hogs and 7 sheep; a hog cost 5 dollars more than a sheep, while the hogs and sheep both cost the same sum; what was the cost of each?

6. William bought 3 oranges and 5 lemons; an orange cost 2 cents more than a lemon, while the oranges and the lemons each cost the same sum; what was the cost of each?

7. William has 10 cents more than Daniel; but 7 times Daniel's money is equal to twice that of William; how many cents has each?

8. The greater of 2 numbers exceeds the less by 14; and 3 times the greater is equal to 10 times the less; what are the numbers?

9. Moses is 16 years younger than his brother Joseph; but 3 times the age of Joseph is equal to 5 times that of Moses; what are their ages?

10. The difference between the ages of a man and his wife is 7 years; and 6 times the age of the man is equal to 8 times the age of his wife; what are their ages?

LESSON XIV.

1. If x represents a certain number, what will represent one half the number?

To divide a number, we draw a line beneath it, under which we place the divisor; thus, to divide 1 by 2, it is written $\frac{1}{2}$, which is read *one half, or one divided by two.* In the same manner, one half of x would be written thus, $\frac{x}{2}$; which may be read *one half of x*, or *x divided by 2.*

In a similar manner, one third of x is written $\frac{x}{3}$; two thirds of x is written $\frac{2x}{3}$.

2. If $\frac{x}{2}$ is equal to 4, what is x equal to?

3. If $\frac{x}{3}$ is equal to 5, what is x equal to?

4. If $\frac{2x}{3}$ is equal to 8, what is x equal to? If *two thirds* of x is equal to 8, *one* third of x is equal to one half of 8, or 4 (since *one* half of *two* thirds is *one* third); and if *one third* of x is equal to 4, x is equal to *three* times 4, or 12.

Or thus: if $2x$ divided by 3 is equal to 8, $2x$ must be equal to 3 times 8, or 24; and if $2x$ is equal to 24, x is equal to one half of 24, or 12.

Either of these methods may be used in finding the value of x in similar expressions.

5. If $\frac{3x}{4}$ is equal to 9, what is x equal to?

6. If $\frac{5x}{8}$ is equal to 10, what is x equal to?

7. If $\frac{7x}{11}$ is equal to 14, what is x equal to?

8. If $\frac{3x}{2}$ is equal to 9, what is x equal to?

9. If $\frac{4x}{3}$ is equal to 12, what is x equal to?

10. If $\frac{5x}{3}$ is equal to 20, what is x equal to?

11. If $\frac{7x}{5}$ is equal to 14, what is x equal to?

12. If $\frac{9x}{7}$ is equal to 18, what is x equal to?

13. What is the sum of x and $\frac{x}{2}$? or of $x+\frac{x}{2}$?

Since x is equal to $\frac{2x}{2}$, we have $x+\frac{x}{2}$ equal to $\frac{2x}{2}+\frac{x}{2}$, which is equal to $\frac{3x}{2}$.

14. What will represent the sum of $2x$ and $\frac{x}{2}$, or of $2x+\frac{x}{2}$?

15. What will represent the sum of $x+\frac{x}{3}$?

16. What will represent the sum of $x+\frac{2x}{3}$? Of $2x+\frac{x}{3}$?

17. What is the sum of $x+\frac{x}{5}$? Of $x+\frac{3x}{5}$? Of $2x+\frac{4x}{5}$?

18. What is the sum of $x+\frac{2x}{7}$? Of $x+\frac{3x}{8}$? Of $2x+\frac{3x}{7}$?

19. There is a certain number, to which if the half of itself be added, the sum will be 15; what is the number?

20. William has half as many cents as Daniel, and they both together have 21; how many cents has each?

21. The age of Mary is one third that of Jane, and the sum of their ages is 24 years; what is the age of each?

22. A pasture contains 44 sheep and cows; the number of cows is one third the number of sheep; how many are there of each?

23. The sum of the ages of Ruth and Eliza is 24 years; while the age of the former is three fifths of that of the latter; what is the age of each?

24. James and John together have 18 cents, and John has four fifths as many as James; how many has each?

25. Two places, A and C, are 40 miles apart; between them is a village which is two thirds as far from C as it is from A; what is its distance from each of the places?

26. The sum of two numbers is 21, and the smaller number is three fourths of the larger; what are the numbers?

27. Thomas and Charles have 35 cents, and Charles has half as many more cents as Thomas; how many cents has each?

28. The double of a certain number, increased by one third of itself, is equal to 21; what is the number?

29. William, James, and Robert, together, have 33 cents; James has twice as many as William, and Robert has one third as many as James; how many cents has each?

30. What number is that, which being increased by its half and its fourth, equals 21?

31. What number is that, which being increased by its half, its fourth, and 4 more, equals 25?

32. A boy, being asked how much money he had, replied, that if one half and one third of his money, and 9 cents more, were added to it, the sum would be 20 cents; how much money had he?

33. There are three numbers, whose sum is 44; the second is equal to one third of the first, and the third is equal to the second and twice the first; what are the numbers?

34. There are four towns in the order of the letters, A, B, C, and D; the distance from B to C is one fifth of the distance from A to B, and the distance from C to D is equal to twice the distance from A to C; the whole distance from A to D is 72 miles. Required the distance from A to B, from B to C, and from C to D.

35. What number is that, to which if its half, its fourth, and 26 more be added, the sum will be equal to 5 times the number?

36. There is a fish whose head is 6 inches long, and the tail is as long as the head and half the body, and the body is as long as the head and tail; what is the length of the whole fish?

37. A gentleman being asked his age, replied, "If to my age you add its half, its third, and 28 years, the sum will be equal to three times my age." Required his age.

☞ **The preceding exercises will serve to give the learner some idea of the nature of Algebra, and of the manner in which it may be applied to the solution of problems. We shall now proceed to consider the subject in a regular and scientific manner.**

ELEMENTS OF ALGEBRA.

CHAPTER I.

PRELIMINARY DEFINITIONS AND PRINCIPLES.

Note to Teachers.—In general, the Introduction, embracing Articles 1 to 15, need not be thoroughly studied until the pupil reviews the book.

Article **1.** In Algebra, numbers and quantities are represented by symbols. These symbols are the letters of the alphabet.

Art. **2.** Quantity is anything that is capable of increase or decrease; such as numbers, lines, space, time, motion, &c.

Art. **3.** Quantity is called *magnitude*, when presented or considered in an undivided form, such as a quantity of water.

Art. **4.** Quantity is called *multitude*, when it is made up of individual and distinct parts, such as three cents, which is a quantity composed of three single cents.

Art. **5.** One of the single parts of which a quantity of multitude is composed, is called the *unit of quantity*, or *measuring unit;* thus, one cent is the *measuring unit* of the quantity three cents. The value or measure of every quantity, is the number of times it contains its measuring unit.

Art. **6.** In quantities of magnitude, where there is no natural unit, it is necessary to fix upon an artificial unit, as a standard of measure; and then to find the value of the quantity, we must ascertain how often it contains its *unit of measure.* Thus, to measure the length of a line, we take a certain assumed distance called a foot, and applying it a certain number of times, say five, we ascertain that the line is five feet long; in this case, one foot is the *unit of measure.*

Art. **7.** The *numerical* value of any quantity, is the number that expresses how many times it contains its unit of measure. Thus, in the preceding example, the line being 5 feet long, its numerical

Review.—1. How are numbers and quantities represented in Algebra? What are symbols? 2. What is a quantity? 3. When is quantity called magnitude? 4. When is quantity called multitude? 5. What is the unit of quantity? 6. How is the value of a quantity ascertained, when there is no natural unit? 7. What is the numerical value of any quantity?

value is 5. The same quantity may have different numerical values, according to the unit of measure that is assumed.

ART. 8. A *unit* is a single or whole thing of an order or kind.

ART. 9. *Number* is an expression denoting a unit, or a collection of units. Numbers are either abstract or concrete.

ART. 10. An *abstract* number denotes how many times a unit is to be taken. A *concrete*, or applicate number, denotes the units that are taken.

Thus, 4 feet is a concrete number; while 4 is an abstract number, which merely shows the number of units that are taken. A concrete number may be defined to be the product of the unit of measure by the corresponding abstract number. Thus, 6 dollars are equal to 1 dollar multiplied by 6, or 1 dollar taken 6 times.

ART. 11. In Algebra, quantities are represented by numbers, and the letters used, stand for numbers.

ART. 12. There are two kinds of questions in Algebra, *theorems* and *problems*.

ART. 13. In a *theorem*, it is required to demonstrate some relation or property of numbers, or abstract quantities.

ART. 14. In a *problem*, it is required to find the value of some unknown number or quantity, by means of certain given relations existing between it and others, which are known.

ART. 15. Algebra is a general method of solving problems and demonstrating theorems, by means of *figures, letters,* and *signs.* The letters and signs are sometimes called symbols.

DEFINITION OF TERMS, AND EXPLANATION OF SIGNS.

ART. 16. *Known quantities* are those whose numerical values are given, or supposed to be known: *unknown quantities* are those whose numerical values are not known.

ART. 17. Known quantities are generally represented by the first letters of the alphabet, as a, b, c, &c.; and unknown quantities by the last letters, as x, y, z.

ART. 18. The following are the principal signs used in Algebra:

$$=,\ +,\ -,\ \times,\ \div,\ (\ ),\ >,\ \surd.$$

Each of these signs is the representative of certain words;

REVIEW.—8. What is a unit? 9. What is number? 10. What does an abstract number denote? What does a concrete number denote? 11. What do the letters used in Algebra represent? 12. How many kinds of questions are there in Algebra? What are they? 13. What is a theorem? 14. What is a problem? 15. What is Algebra? 16. What are known quantities? What are unknown quantities? 17. By what are known quantities represented? By what are unknown quantities represented? 18. Write on a slate, or a blackboard, the principal signs used in Algebra. What do the signs represent? For what purpose are they used?

they are used for the purpose of expressing the various operations, in the most clear and brief manner.

ART. **19.** The sign of *equality*, $=$, is read *equal to*. It denotes that the quantities between which it is placed are equal to each other. Thus, $a=3$, denotes that the quantity represented by a is equal to 3.

ART. **20.** The sign of *addition*, $+$, is read *plus*. It denotes that the quantity to which it is prefixed, is to be added to some other quantity.

Thus, $a+b$ denotes that b is to be added to a. If $a=2$ and $b=3$, then $a+b=2+3$, which are $=5$.

ART. **21.** The sign of *subtraction*, $-$, is read *minus*. It denotes that the quantity to which it is prefixed is to be subtracted. Thus, $a-b$ denotes that b is to be subtracted from a. If $a=5$ and $b=3$ then $5-3=2$.

ART. **22.** The signs $+$ and $-$ are called *the* signs; the former is called the *positive*, and the latter the *negative* sign; they are said to be *contrary* or *opposite*.

ART. **23.** Every quantity is supposed to be preceded by one or the other of these signs. Quantities having the positive sign are called *positive;* and those having the negative sign are called *negative.* When a quantity has no sign prefixed to it, it is considered positive.

ART. **24.** Quantities having the same sign are said to have *like* signs; those having different signs are said to have *unlike* signs. Thus, $+a$ and $+b$, or $-a$ and $-b$ have like signs; while $+c$ and $-d$ have unlike signs.

ART. **25.** The sign of *multiplication*, $\times$, is read *into*, or *multiplied by*. It denotes that the quantities between which it is placed, are to be multiplied together.

A dot or point is sometimes used instead of the sign $\times$. Thus, $a\times b$ and $a.b$, both mean that b is to be multiplied by a. The dot is not used to denote the multiplication of figures, because it is used to separate whole numbers and decimals.

The product of two or more letters is generally denoted by writing them in close succession. Thus, ab denotes the same as $a\times b$, or $a.b$; and abc means the same as $a\times b\times c$, or $a.b.c$.

REVIEW.—19. How is the sign of equality, $=$, read? What does it denote? 20. How is the sign $+$ read? What does it denote? 21. How is the sign $-$ read? What does it denote? 22. What are the signs plus and minus called, by way of distinction? Which is positive, and which negative? 23. When quantities are preceded by the sign plus, what are they said to be? By the sign minus? When a quantity has no sign prefixed, what sign is understood? 24. When do quantities have like signs? When unlike signs? 25. How is the sign $\times$ read, and what does it denote? What other methods are there of representing multiplication, besides the sign $\times$?

ART. **26.** Quantities that are to be multiplied together, are called *factors*. The *continued product* of several factors, means that the product of the first and second is to be multiplied by the third, this product by the fourth, and so on. Thus, the continued product of a, b, and c, is expressed by $a\times b\times c$, or abc.

If $a=2$, $b=3$, and $c=5$, then $abc=2\times3\times5=6\times5=30$.

ART. **27.** The sign of *division*, $\div$, is read *divided by*. It denotes that the quantity preceding it is to be divided by that following it. The division of two quantities is more frequently represented, by placing the dividend as the numerator, and the divisor as the denominator of a fraction. Thus, $a\div b$, or $\frac{a}{b}$, means, that a is to be divided by b. If $a=12$ and $b=3$, then $a\div b=12\div3=4$; or $\frac{a}{b}=\frac{12}{3}=4$.

Division is also represented thus, $a\underline{|b}$, where a denotes the dividend, and b the divisor.

ART. **28.** The sign $>$, is called the sign of *inequality*. It denotes that one of the two quantities between which it is placed, is greater than the other, the *opening* of the sign being turned towards the *greater* quantity.

Thus, $a>b$ denotes that a is greater than b. It is read, *a greater than b*. If $a=5$, and $b=3$, then $5>3$.

Also, $c<d$ denotes that c is less than d. It is read, *c less than d*. If $c=4$ and $d=7$, then $4<7$.

ART. **29.** The sign ∞, denotes a quantity greater than any that can be assigned; that is, a quantity indefinitely great, or infinity.

ART. **30.** The *numeral coëfficient* of a quantity is a number prefixed to it, to show how often the quantity is to be taken. Thus, if the quantity represented by a is to be added to itself several times, as $a+a+a+a$, we write it but once, and place a number before it, to show how often it is taken.

Thus, $a+a+a+a=4a$; and $ax+ax+ax=3ax$.

ART. **31.** The *literal coëfficient* of a quantity, is a quantity by which it is multiplied. Thus, in the quantity ay, a may be considered the coëfficient of y, or y may be considered the coëfficient of a. The literal coëfficient is generally regarded as a known quantity.

REVIEW.—26. What are factors? How many factors in a? In ab? In abc? In $5abc$? 27. How is the sign $\div$ read, and what does it denote? What other methods are there of representing the division of two quantities? 28. What does the sign of inequality, $>$, denote? Which quantity is placed at the opening? 29. What does the sign ∞ denote? 30. What is the numeral coëfficient of a quantity? How often is ax taken in the expression $3ax$? In $5ax$? In $7ax$? 31. What is the literal coëfficient of a quantity?

ART. **32.** The coëfficient of a quantity may consist of a number, and also of a literal part. Thus, in the quantity $5ax$, $5a$ may be regarded as the coëfficient of x. If $a=2$, then $5a=10$, and $5ax=10x$.

When no numeral coëfficient is prefixed to a quantity, its coëfficient is understood to be unity. Thus, a is the same as $1a$, and bx is the same as $1bx$.

ART. **33.** The *power* of a quantity is the product arising from multiplying the quantity by itself one or more times. When the quantity is taken twice as a factor, the product is called its *square*, or *second* power; when three times, the *cube*, or *third* power; when four times, the *fourth* power, and so on.

Thus, $a\times a=aa$, is the *second* power of a; $a\times a\times a=aaa$, is the *third* power of a; $a\times a\times a\times a=aaaa$, is the *fourth* power of a.

Instead of repeating the same quantity as a factor, a small figure, called an *exponent*, is placed to the right, and a little above it, to point out the number of times the quantity is taken as a factor. Thus, aa is written a^2; aaa is written a^3; $aaaa$ is written a^4; $aabbb$ is written a^2b^3.

When a letter has no exponent, it is considered to be the *first*, or *simple* power of the quantity, and unity is considered to be its exponent. Thus, a is the same as a^1, each expressing the first power of a.

ART. **34.** To involve or raise a quantity to any given power, is to find that power of the quantity.

ART. **35.** The *root* of any quantity is another quantity, some power of which is equal to the given quantity. The root is called the square root, cube root, fourth root, &c., according to the number of times it must be taken as a factor to produce the given quantity.

Thus, since $a\times a=a^2$, therefore a is the second root, or square root of a^2. In the same manner, x is the third root, or cube root of x^3, since $x\times x\times x=x^3$.

ART. **36.** To *extract* any root of a quantity, is to find that root.

ART. **37.** The sign $\sqrt{}$, is called the radical sign. When placed before a quantity it indicates that its root is to be extracted.

Thus $\sqrt[2]{a}$, or $\sqrt{a}$, denotes the square root of a; $\sqrt[3]{a}$, denotes the cube root of a; $\sqrt[4]{a}$ denotes the fourth root of a.

ART. **38.** The number placed over the radical sign is called the *index* of the root. Thus, 2 is the index of the square root, 3 of the

REVIEW.—32. When a quantity has no coëfficient written, what coëfficient is understood? 33. What is the power of a quantity? What is meant by the second power of a? By the third power of a? What is an exponent? For what is it used? How many times is x taken as a factor in x^2? In x^3? In x^5? Where no exponent is written, what exponent is understood? 35. What is the root of a quantity? 37. What is the sign $\sqrt{}$ called, and what does it denote?

cube root, 4 of the fourth root, and so on. When the radical has no index over it, 2 is understood.

Thus, $\sqrt{9}=3$, $\sqrt[3]{8}=2$, $\sqrt[4]{16}=2$.

ART. **39.** Every quantity written in algebraic language, that is, by means of algebraic symbols, is called an *algebraic quantity*, or an *algebraic expression*. Thus,

$3a$ is the algebraic expression of 3 times the number a; $3a-4b$, is the algebraic expression for 3 times the number a, diminished by 4 times the number b; $2a^2+3ab$, is the algebraic expression for twice the square of a, increased by 3 times the product of the number a by the number b.

ART. **40.** An algebraic quantity not united to any other by the sign of addition or subtraction, is called a *monomial*, or a quantity of one term, or simply a *term*. A monomial is sometimes called a *simple quantity*. Thus, a, $3a$, $-a^2b$, $2any^2$, are monomials, or simple quantities.

ART. **41.** An algebraic expression composed of two or more terms, is called a *polynomial*, or a *compound quantity*.

Thus, $c+2d-b$ is a polynomial.

ART. **42.** A polynomial composed of two terms, is called a *binomial*. Thus, $a+b$, $a-b$, and c^2-d are binomials.

A binomial, in which the second term is negative, as $a-b$, is sometimes called a *residual quantity*.

ART. **43.** A polynomial consisting of *three terms*, is called a *trinomial*. Thus, $a+b+c$, and $a-b-c$ are trinomials.

ART. **44.** The *numerical value* of an algebraic expression is the number obtained, by giving particular values to the letters, and then performing the operations indicated.

In the algebraic expression $2a+3b$, if $a=4$, and $b=5$, then $2a=8$, and $3b=15$, and the numerical value is $8+15=23$.

ART. **45.** The value of a polynomial is not affected by changing the order of the terms, provided each term retains its respective sign. Thus, a^2+2a+b is the same as $b+a^2+2a$. This is self-evident.

ART. **46.** Each of the *literal* factors of any simple quantity or term is called a *dimension* of that term; and the *degree* of any term depends on the number of its literal factors.

Thus, ax consists of two literal factors, a and x, and is of the *second degree*. The quantity a^2b contains three literal factors, a, a,

REVIEW.—38. What is the number placed over the radical sign called? 39. What is an algebraic quantity? 40. What is a monomial? What is a simple quantity? 41. What is a polynomial? 42. A binomial? A residual quantity? 43. A trinomial? 44. What is meant by the numerical value of an algebraic expression?

and b, and is of the *third degree.* $2a^3x^2$ contains five literal factors, a, a, a, x, and x, and is of the *fifth degree;* and so on.

ART. **47.** A polynomial is said to be *homogeneous,* when each of its terms is of the same degree.

Thus, the quantity $2a-3b+c$ is of the first degree, and homogeneous; $a^2+3bc+xy$, is of the second degree, and homogeneous; x^3-8ay^2 is of the third degree, and homogeneous; a^3+x^2 is not homogeneous.

ART. **48.** A *parenthesis,* ( ), is used to show that all the terms of a compound quantity are to be considered together as a single term.

Thus, $4(a-b)$ means that $a-b$ is to be multiplied by 4; $(a+x)(a-x)$ means that $a+x$ is to be multiplied by $a-x$; $10-(a+c)$ means that $a+c$ is to be subtracted from 10; $(a-b)^2$ means that $a-b$ is to be raised to the second power; and so on.

ART. **49.** A *vinculum,* ——, is sometimes used instead of a parenthesis. Thus, $\overline{a-b}\times x$ means the same as $(a-b)x$. Sometimes the vinculum is placed vertically, it is then called a *bar.*

Thus, $\left.\begin{array}{r} a \\ -x \\ +4 \end{array}\right| y^2$ has the same meaning as $(a-x+4)y^2$.

ART. **50.** *Similar,* or *like* quantities are those composed of the same letters, affected with the same exponents. Thus, $7ab$ and $-3ab$, also $4a^3b^2$ and $7a^3b^2$, are similar terms. The quantities $2a^2b$ and $2ab^2$ are not similar, for, though they are composed of the same letters, yet these letters have different exponents.

ART. **51.** The *reciprocal* of a quantity, is unity divided by that quantity. Thus, the reciprocal of 2 is $\frac{1}{2}$, and of a is $\frac{1}{a}$.

ART. **52.** The same letter accented, is often used to denote quantities which occupy similar positions in different equations or investigations. Thus, a, a', a'', a''', represent four different quantities; of which a' is read, a prime; a'' is read, a second; a''' is read, a third, and so on.

EXAMPLES.

The following examples are intended to exercise the learner in the use and meaning of the signs.

REVIEW.—46. What is the dimension of a term? On what does the degree of a term depend? What is the degree of the term xy? Of xyz? Of $2axy$? 47. When is a polynomial said to be homogeneous? 48. For what is a parenthesis used? 49. What is a vinculum, and for what is it used? 50. What are similar, or like quantities? 51. What is the reciprocal of a quantity? 52. When a letter, as a', has one accent, what does it represent, and how is it read? How is a with two accents read?

Let the pupil copy each example on his slate, or on the black board, and then express it in common language.

Also, let the numerical value of each expression be found, on the supposition that $a=4$, $b=3$, $c=5$, $d=10$, $x=2$, and $y=6$.

1. $c+d-b$. . . . Ans. 12.
2. $4a-x$. . . . . Ans. 14.
3. $-3ax$. . . . Ans. -24.
4. $6a^2x$. . . . Ans. 192.
5. $\frac{a+c}{b}$. . . . . Ans. 3.
6. $\frac{b+c+d-x}{a}$. . . . Ans. 4
7. $\frac{ay}{b}+\frac{cd}{x}$. . . . . Ans. 33
8. $3a^2+2cx-b^3$. . . Ans. 41
9. $a(a+b)$. . . . . Ans. 28.
10. $a+b\times a-b$. . . . . Ans. 13.
11. $(a+b)(a-b)$. . . . . Ans. 7.
12. $x^2-3(a+x)(a-x)+2by$. . . . . Ans. 4.
13. $3ax-\frac{2(a-x)}{3(a+x)}-4\sqrt{2ax}$. . . . . Ans. $7\frac{7}{9}$.
14. $\frac{2ax^2}{(a-x)^2}-6x\sqrt{a}$. . . . . Ans. -16.

In case further exercises should be required to teach the pupils the use of the signs, the following equivalent expressions may be employed, in which each letter may have any value whatever, provided that the same value be attributed to the same letter throughout the same expression.

15. $3(a+c)(a-c)=3a^2-3c^2$.
16. $5(a-b)^2=5a^2-10ab+5b^2$.
17. $\frac{a^2-x^2}{a+x}=a-x$.
18. $\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3$.

Examples in which words are to be converted into algebraic symbols.

1. Three times a, plus b, minus four times c. Or, three into a, plus b, minus 4 into c.
2. Five times a, divided by three times b.
3. a minus b, into three times c.
4. a, minus three times b into c.
5. a plus b, divided by three c.
6. a, plus b divided by three c.
7. 5 into a minus three into b, divided by c minus d.
8. a squared, minus three a into b, plus 5 times c into d squared.
9. x cubed minus b cubed, divided by x squared minus b squared.
10. Five a squared, into a plus b, into c minus d, minus three times x fourth power.
11. a fifth power minus b fifth power, divided by a minus b, raised to the fifth power.

12. a squared plus b squared, divided by a plus b, squared.
13. The square root of a, minus the square root of x
14. The square root of a, minus x.
15. The square root of a minus x.
16. The square root of b squared minus four into a into c.

ANSWERS.

1. $3a+b-4c$.
2. $\frac{5a}{3b}$.
3. $(a-b)3c$.
4. $a-3bc$.
5. $\frac{a+b}{3c}$.
6. $a+\frac{b}{3c}$.
7. $\frac{5a-3b}{c-d}$.
8. $a^2-3ab+5cd^2$.
9. $\frac{x^3-b^3}{x^2-b^2}$.
10. $5a^2(a+b)(c-d)-3x^4$.
11. $\frac{a^5-b^5}{(a-b)^5}$.
12. $\frac{a^2+b^2}{(a+b)^2}$.
13. $\sqrt{a}-\sqrt{x}$.
14. $\sqrt{a}-x$.
15. $\sqrt{(a-x)}$.
16. $\sqrt{(b^2-4ac)}$.

ADDITION.

ART. **53.** ADDITION in Algebra, is the process of collecting two or more algebraic quantities into one expression, called their sum.

CASE I.

When the quantities are similar, and have the same sign.

1. James has 3 pockets, each containing apples; in the first he has 3 apples, in the second 4 apples, and in third 5 apples.

In order to find how many apples he has, suppose he proceeds to find their sum in the following manner:

3 *apples,*
4 *apples,*
5 *apples,*
12 *apples.*

Suppose, however, that, instead of writing the word *apples*, he should merely use the letter a, thus:

$$\begin{array}{r} 3a \\ 4a \\ 5a \\ \hline 12a \end{array}$$

REVIEW.—53. What is algebraic addition? When quantities are similar, and have the same sign, how are they added together?

It is evident that the sum of 3 times a, 4 times a, and 5 times a, would be 12 times a, or $12a$, whatever a might represent.

2. In the same manner the sum of $-3a$, $-4a$, and $-5a$ would be $-12a$.

$$\begin{array}{r} -3a \\ -4a \\ -5a \\ \hline -12a \end{array}$$

Hence, the

RULE,

FOR ADDING SIMILAR QUANTITIES WITH LIKE SIGNS.

Add together the coëfficients of the several quantities, and to their sum annex the common letter, or letters, prefixing the common sign.

NOTE 1.—Let the pupil be reminded, that when a quantity has no coëfficient prefixed, 1 is understood; thus, a is the same as $1a$.

NOTE 2.—Let the pupil also be reminded, that the sum of any number of quantities is the same, in whatever order they are taken. This is self-evident; but it may be illustrated by numbers in the following manner. Suppose it is required to find the sum of the numbers 16, 25, and 34; in adding these numbers together, they may be written in six different ways, in each of which the sum is the same. Thus:

$$\begin{array}{cccccc} 16 & 16 & 25 & 25 & 34 & 34 \\ 25 & 34 & 16 & 34 & 16 & 25 \\ 34 & 25 & 34 & 16 & 25 & 16 \\ \hline 75 & 75 & 75 & 75 & 75 & 75 \end{array}$$

EXAMPLES.

$$\begin{array}{rrrr} 3. & 4. & 5. & 6. \\ 3a & -6xy & 2a^2 & -3a^2b \\ 2a & -xy & 3a^2 & -4a^2b \\ a & -4xy & 5a^2 & -5a^2b \\ 5a & -3xy & 7a^2 & -2a^2b \\ \hline \text{Sum} = 11a & -14xy & 17a^2 & -14a^2b \end{array}$$

In the third example, we will suppose $a=2$, then $3a=3\times2=6$, $2a=2\times2=4$, $a=2$, $5a=5\times2=10$; their sum is $6+4+2+10=22$.

But the sum, 22, is more easily found from the algebraic sum, $11a$, for $11a=11\times2=22$.

In the fourth example, let $x=3$ and $y=2$, and the value of its terms will be

$$\begin{array}{rl} 6xy= & 6\times3\times2=36 \\ xy= & 3\times2=\ \ 6 \\ 4xy= & 4\times3\times2=24 \\ 3xy= & 3\times3\times2=18 \\ \hline \text{the sum of their values is} & =84 \end{array}$$

But this sum is more easily found from the algebraic sum; for

REVIEW.—When several quantities are to be added together, is the result affected by the order in which they are taken?

$14xy = 14 \times 3 \times 2 = 84$. As all these terms are negative, their sum is -84.

In the fifth example let a represent 3 feet, then

$$\begin{array}{lrl} & 2a^2 = 2aa = 2\times3\times3 = 18 & \text{square feet,} \\ & 3a^2 = 3aa = 3\times3\times3 = 27 & \text{" "} \\ & 5a^2 = 5aa = 5\times3\times3 = 45 & \text{" "} \\ & 7a^2 = 7aa = 7\times3\times3 = 63 & \text{" "} \\ \text{and their sum is} & \overline{153} & \text{" "} \end{array}$$

Or the sum $= 17a^2 = 17\times3\times3 = 153$ square feet.

NOTE.—It is recommended to the learner, thus to exemplify the examples numerically, by assigning certain values to the letters; observing throughout each example, to adhere to the same numerical value for the same letter.

What is the sum

7. Of $3b$, $5b$, $7b$, and $9b$? Ans. $24b$.
8. Of $2ab$, $5ab$, $8ab$, and $11ab$? Ans. $26ab$.
9. Of abc, $3abc$, $7abc$, and $12abc$? Ans. $23abc$.
10. Of $5a$ dollars, $8a$ dollars, $11a$ dollars, and $13a$ dollars? Ans. $37a$ dollars.
11. Of $-3ax$, $-5ax$, $-7ax$, and $-4ax$? Ans. $-19ax$.
12. Of $-by$, $-2by$, $-5by$, and $-8by$? Ans. $-16by$.

13.	14.	15.
$3ay+7$	$8x-4y$	$3a^2-2ax$
$ay+8$	$5x-3y$	$5a^2-3ax$
$2ay+4$	$7x-6y$	$7a^2-5ax$
$5ay+6$	$6x-2y$	$4a^2-4ax$

CASE II.

ART. 54. *When quantities are alike, but have unlike signs.*

1. If James receives from one man 6 cents, from another 9 cents, and from a third 10 cents; and then spends, for candy 4 cents, and for apples 3 cents, how much money will he have left?

If the quantities he received be considered positive, then those he spent may be considered negative; and the question is, to find the sum of $+6c$, $+9c$, $+10c$, $-4c$ and $-3c$, which may be written thus:

$$\begin{array}{r} +6c \\ +9c \\ +10c \\ -4c \\ -3c \\ \hline +18c \end{array}$$

Here, it is evident, the true result will be found, by adding the positive quantities into one sum, and the negative quantities into another, and then taking their difference. It is thus found that he received $25c$, and spent $7c$, which left $18c$.

2. Suppose James should receive 5 cents, and then spend 7 cents, what sum would he have left?

If we denote the $5c$ as positive, the $7c$ will be negative, and it is required to find the sum of $+5c$ and $-7c$.

In its present form, however, it is evident that the question is impossible. But if we suppose that James had a certain sum of money before he received the $5c$, we may inquire how much *less* money he had after the operation, than before it; or, in other words, what effect the operation had upon his money. The answer, it is obvious, would be, that his money was *diminished* 2 cents; this would be indicated by the sum of $+5c$ and $-7c$, being $-2c$.

It is thus we say, that the sum of a positive and negative quantity is equal to the *difference* between the two; the object being to find what the *united effect* of the two will be upon some third quantity. This may be further illustrated by the following example.

3. A merchant has a certain capital; during the year it is *increased* by $3a$ and $8a$ dollars, and *diminished* by $2a$ and $5a$ dollars: how much will his capital be increased or diminished at the close of the year?

If we denote the *gains* as positive, the *losses* will be negative. The sum of $+3a$, $+8a$, $-2a$, and $-5a$ is $11a-7a$, which is equal to $+4a$. Hence, we say, that the merchant's capital will be *increased* by $4a$ dollars; and whatever the capital may have been, the result will be the same to increase it by $4a$ dollars, as first to increase it by $3a$ and $8a$ dollars, and then to diminish it by $2a$ and $5a$ dollars. Had the loss been greater than the gain, the effect would be to *diminish* the capital; and this would be indicated, by the sum of the gains and losses being *negative.*

If the gain and loss were equal, it is evident the capital would neither be increased nor diminished; or, in other words, if the amount of the positive quantities was equal to that of the negative, their sum would be 0. Thus, $+3a-3a=0$. If $a=4$, $+3a=+12$ and $-3a=-12$, and $+12-12=0$.

From this the pupil will perceive, that to add a negative quantity is the same as to subtract a positive quantity. In such cases, the process of addition is called *algebraic addition,* and the sum is called the *algebraic sum,* to distinguish them from arithmetical addition, and arithmetical sum. Hence, the

RULE,

FOR THE ADDITION OF QUANTITIES WHICH ARE ALIKE, BUT HAVE UNLIKE SIGNS.

Find the sum of the coëfficients of the similar positive quantities; also, the sum of the coëfficients of the similar negative quantities. Subtract the less sum from the greater; then, to the difference prefix the sign of the greater, and annex the common literal part.

4. What is the sum of $+3a$, $-5a$, $+9a$, $-6a$, and $+7a$?

Here, the sum of the coëfficients of the positive terms, is

$$3+9+7=+19$$

The sum of the coëfficients of the negative terms, is

$$-5-6=-11$$

The difference between 19 and 11 is 8, to which, prefixing the sign of the greater, and annexing the literal part, we have for the required sum $+8a$.

In practice, it is most convenient to write the different terms under each other. Thus,

$$\begin{array}{r} 3a \\ -5a \\ 9a \\ -6a \\ 7a \\ \hline \text{Sum}=8a \end{array}$$

Beginners, however, will sometimes find it easier to arrange the positive quantities in one column, and the negative in another. The preceding example may be arranged as in the margin.

$$\begin{array}{l} 3a-5a \\ 9a-6a \\ 7a \\ \hline 19a-11a=8a \end{array}$$

EXAMPLES.

5. What is the sum of $8a$ and $-5a$? Ans. $3a$.

6. What is the sum of $5a$ and $-8a$? Ans. $-3a$.

7. What is the sum of $-7ax$, $3ax$, $6ax$, and $-ax$? Ans. ax.

8. What is the sum of $5abx$, $-7abx$, $3abx$, $-abx$, and $4abx$? Ans. $4abx$.

9. Add together, $4ac$, $5ac$, $-3ac$, $7ac$, $-6ac$, $-2ac$, $9ac$, and $-17ac$. Ans. $-3ac$.

10. Find the sum of $6a-4b$, $3a+2b$, $-7a-8b$, and $-a+9b$. Ans. $a-b$.

11. Find the sum of $8ax-2by$, $-2ax+3by$, $3ax-4by$, and $-9ax+8by$. Ans. $5by$.

12. Find the sum of $3ab-10x$, $-3ab+7x$, $3ab-6x$, $-ab+2x$, and $-2ab+7x$. Ans. 0.

13. Find the sum of $4a^2-2b$, $-6a^2+2b$, $2a^2-3b$, $-5a^2-8b$, and $-3a^2+9b$. Ans. $-8a^2-2b$.

14. Find the sum of $xy-ac$, $3xy-9ac$, $-7xy+5ac$, $4xy+6ac$, and $-xy-2ac$. Ans. $-ac$.

NOTE.—The operation of collecting the similar terms in any algebraic expression into one sum, as exemplified in this case, is sometimes called the *Reduction of Polynomials*. The following are examples.

15. Reduce $3ab+5c-7ab+8c+8ab-14c-2ab+c$ to its simplest form. Ans. $2ab$.

REVIEW.—54. How are quantities added together that are similar, but have unlike signs?

16. Reduce $5a^2c-3b^2+4a^2c+5b^2-8a^2c+2b^2$ to its simplest form. Ans. a^2c+4b^2

CASE III.

ART. **55.** *When the quantities are unlike, or partly like and partly unlike.*

1. Thomas has a marbles in one hand, and b marbles in the other; what expression will represent the number in both? If a is represented by 3, and b by 4, then the number in both would be represented by $3+4$, or 7.

In the same manner, the number in both would be represented by $a+b$; but unless the numerical values of a and b are given, it is evidently impossible to represent their sum more concisely, than by $a+b$.

In the same manner, the sum of the quantities $a+b$ and $c+d$, is represented by $a+b+c+d$.

If, in any expression, there are two or more like quantities, it is obvious, that they may be reduced to a single expression by the preceding rules. Thus, the sum of $2a+x$ and $3a+y$, is equal to $2a+3a+x+y$, which reduces to $5a+x+y$.

It is evident that this case embraces the two preceding cases; hence, the

GENERAL RULE,

FOR THE ADDITION OF ALGEBRAIC QUANTITIES.

Write the quantities to be added, placing those that are similar under each other; then reduce the similar terms, and annex the other terms with their proper signs.

REMARK.—If a reason is asked for placing similar terms under each other, the reply is, that it is not absolutely necessary; but as we can only add similar terms together, *it is a matter of convenience*, to place them under each other.

EXAMPLES.

Add together

2. $6a-4c+3b$, and $-2a-3c-5b$. Ans. $4a-7c-2b$.
3. $2ab+c$, $4ax-2c+14$, $12-2ax$, and $6ab+3c-x$. Ans. $8ab+2ax+2c+26-x$.
4. $14a+x$, $13b-y$, $-11a+2y$, and $-2a-12b+z$. Ans. $a+b+x+y+z$.
5. $a-b$, $2b-c$, $2c-d$, $2d-e$, and $2e+f$. Ans. $a+b+c+d+e+f$.
6. $-7b+3c$, $4b-2c+3x$, $3b-3c$, and $2c-2x$. Ans. x.
7. $3(a+b)$, $5(a+b)$, and $7(a+b)$. Ans. $15(a+b)$.

REVIEW.—55. What is the general rule for the addition of algebraic quantities? In writing them, why are similar quantities placed under each other?

NOTE.—The learner should be reminded, that the quantities in the parentheses are to be considered as one quantity; then it is evident, that 3 times, 5 times, and 7 times *any quantity whatever*, will be equal to 15 times that quantity.

Add together

8. $3a(b+x)$, $5a(b+x)$, $7a(b+x)$, and $-11a(b+x)$.
Ans. $4a(b+x)$.

9. $2c(a^2-b^2)$, $-3c(a^2-b^2)$, $6c(a^2-b^2)$, and $-4c(a^2-b^2)$.
Ans. $c(a^2-b^2)$.

10. $3az-4by-8$, $-2az+5by+6$, $5az+6by-7$, and $-8az-7by+5$.
Ans. $-2az-4$.

11. $8ax-3cz^2$, $-5ax+5cz^2$, $ax+2cz^2$, and $-4ax-4cz^2$. Ans. 0.

12. $8a+b$, $2a-b+c$, $-3a+5b+2d$, $-6b-3c+3d$, and $-5a+7c-2d$.
Ans. $2a-b+5c+3d$.

13. $7x-6y+5z+3-g$, $-x-3y-8-g$, $-x+y-3z-1+7g$, $-2x+3y+3z-1-g$, and $x+8y-5z+9+g$.
Ans. $4x+3y+2+5g$.

14. $2a^2+5ab-xy$, $-7a^2+3ab-3xy$, $-3a^2-7ab+5xy$, and $9a^2-ab-2xy$.
Ans. a^2-xy.

15. $5a^3b^2-8a^2b^3+x^2y+xy^2$, $4a^2b^3-7a^3b^2-3xy^2+6x^2y$, $3a^3b^2+3a^2b^3-3x^2y+5xy^2$, and $2a^2b^3-a^3b^2-3x^2y-3xy^2$. Ans. $a^2b^3+x^2y$.

SUBTRACTION.

ART. **56.** SUBTRACTION in Algebra, is the process of finding the simplest expression for the difference between two algebraic quantities.

In Algebra, as in Arithmetic, the quantity to be subtracted is called the *subtrahend*. The quantity from which the subtraction is to be made, is called the *minuend*. The quantity left, after the subtraction is performed, is called the *difference*, or *remainder*.

REMARK.—The word subtrahend means, *to be subtracted;* the word minuend, *to be diminished*.

1. Thomas has $5a$ cents; if he give $2a$ cents to his brother, how many will he have left?

Since 5 times any quantity, diminished by 2 times the same quantity, leaves 3 times the quantity, the answer is evidently $3a$; that is $5a-2a=3a$.

Hence, to find the difference between two positive similar quantities, *we find the difference between their coëfficients, and prefix it to the common letter, or letters.*

Let it be noted, that the sign of the quantity to be subtracted, is changed from *plus* to *minus*.

	2.	3.	4.	5.
From	$5x$	$7ab$	$8xy$	$11a^2x$
Take	$3x$	$3ab$	$5xy$	$5a^2x$
Remainder	$2x$	$4ab$	$3xy$	$6a^2x$

6. From $9a$, take $4a$. . . . . Ans. $5a$.
7. From $11b$, take $11b$. . . . . Ans. 0.
8. From $11axy$, take $3axy$. . . . . Ans. $8axy$.
9. From $12bcx$, take $5bcx$. . . . . Ans. $7bcx$.
10. From $13hmp$, take $9hmp$. . . . . Ans. $4hmp$.
11. From $3a^2$, take $2a^2$. . . . . Ans. a^2.
12. From $7b^2xy$, take $4b^2xy$. . . . . Ans. $3b^2xy$.

ART. **57.**—1. Thomas has a number of apples, represented by a; if he give away a quantity, represented by b, what expression will represent the number of apples he has left?

If a represents 6, and b 4, then the number left would be represented by $6-4$, which is equal to 2; and whatever numbers a and b represent, it is evident that their difference may be expressed in the same way, that is, by $a-b$.

Hence, *to find the difference between two quantities that are not similar, we place the sign minus before the quantity that is to be subtracted.*

Let the pupil here notice again, that the sign of the quantity to be subtracted, is changed from *plus* to *minus*.

2. From c, take d. . . . . Ans. $c-d$.
3. From $2m$, take $3n$. . . . . Ans. $2m-3n$.
4. From $5b$, take $3c$. . . . . Ans. $5b-3c$.
5. From ab, take cd. . . . . Ans. $ab-cd$.
6. From a^2x, take ax^2. . . . . Ans. a^2x-ax^2.
7. From x^2, take x. . . . . Ans. x^2-x.
8. From xy, take yz. . . . . Ans. $xy-yz$.

ART. **58.**—1. Let it be required to subtract $5+3$ from 9.

If we subtract 5 from 9, the remainder will be $9-5$; but we wish to subtract, not only 5, but also 3; hence, after we have subtracted 5, we must also subtract 3; this gives for the remainder, $9-5-3$, which is equal to 1.

REVIEW.—56. What is Subtraction in Algebra? What is the quantity to be subtracted, called? What is the quantity called, from which the subtraction is to be made? What does subtrahend mean? What does minuend mean? How do you find the difference between two positive similar quantities? 57. How do you find the difference between two quantities that are not similar?

2. Again, suppose that it is required to subtract 5—3 from 9. If we subtract 5 from 9, the remainder will be 9—5; but the quantity to be subtracted is 3 less than 5, and we have, therefore, subtracted 3 too much; we must, therefore, add 3 to 9—5, which gives for the true remainder, 9—5+3, which is equal to 7.

3. Let it now be required to subtract $b-c$ from a.

If we take b from a, the remainder is $a-b$; but, in doing this, we have subtracted c too much; hence, to obtain the true result, we must add c. This gives, for the true remainder, $a-b+c$.

If $a=9$, $b=5$, and $c=3$, the operation and illustration by figures would stand thus:

	from a	from	9	$=9$
	take $\underline{b-c}$	take	$\underline{5-3}$	$\underline{=2}$
Remainder,	$a-b+c$	Rem.	$9-5+3$	$=7$

The same principle may be further illustrated by the following examples.

4. $a-(c-a)=a-c+a=2a-c$.
$a-(a-c)=a-a+c=c$.
$a+b-(a-b)=a+b-a+b=2b$.

Let it be noted, that in the result in each of the preceding examples, the signs of the quantity to be subtracted have been changed from *plus* to *minus*, and from *minus* to *plus;* hence, in order to subtract a quantity, it is merely necessary to change the signs and add it. Hence, the

RULE,

FOR FINDING THE DIFFERENCE BETWEEN TWO ALGEBRAIC QUANTITIES.

Write the quantity to be subtracted under that from which it is to be taken, placing similar terms under each other. Conceive the signs of all the terms of the subtrahend to be changed, and then reduce the result to its simplest form.

NOTE.—It is a good plan with beginners, to direct them to write the example a second time, and then actually change the signs, and add, as in the following example. They should do this, however, only till they become familiar with the rule.

From	$5a+3b-c$	The same, with the	$5a+3b-c$
Take	$\underline{2a-2b-3c}$	signs of the subtra-	$\underline{-2a+2b+3c}$
Remain.	$3a+5b+2c$	hend changed.	$3a+5b+2c$

EXAMPLES.

	6.	7.	8.
From	$3ax-2y$	$4cx^2-3by^2$	$8xyz+3az-8$
Take	$\underline{2ax+3y}$	$\underline{2cx-3by^2}$	$\underline{5xyz-3az+8}$
Remainder,	$ax-5y$	$4cx^2-2cx$	$3xyz+6az-16$

	9.	10.	11.
From	$7x+4y$	$3a-2b$	$6ax-4y^2+3$
Take	$6x-y$	$5a-3b$	$3ax-6y^2+2$

12. From 14, take $ab-5$. . . . . Ans. $19-ab$.
13. From $a+b$, take a. . . . . Ans. b.
14. From a, take $a+b$. . . . . Ans. $-b$.
15. From x, take $x-5$. . . . . Ans. 5.
16. From $3ax$, take $2ax+7$. . . . . Ans. $ax-7$.
17. From $x+y$, take $x-y$. . . . . Ans. $2y$.
18. From $x-y$, take $x+y$. . . . . Ans. $-2y$.
19. From $x-y$, take $y-x$. . . . . Ans. $2x-2y$.
20. From $x+y+z$, take $x-y-z$. . . . . Ans. $2y+2z$.
21. From $5x+3y-z$, take $4x+3y+z$. . . . . . . Ans. $x-2z$.
22. From a, take $-a$. . . . . Ans. $2a$.
23. From $8a$, take $-3a$. . . . . Ans. $11a$.
24. From a, take $-4a$. . . . . Ans. $5a$.
25. From $5b$, take $11b$. . . . . Ans. $-6b$
26. From a, take $-b$ . . . . Ans. $a+b$.
27. From $3a$, take $-2b$. . . . . Ans. $3a+2b$.
28. From $-9a$, take $3a$. . . . . Ans. $-12a$.
29. From $-7a$, take $-7a$. . . . . Ans. 0.
30. From $-19a$, take $-20a$. . . . . Ans. a.
31. From $-6a$, take $-5a$. . . . . Ans. $-a$.
32. From $-3a$, take $-5b$. . . . . . Ans. $-3a+5b$, or $5b-3a$.
33. From -13, take 3. . . . . Ans. -16.
34. From -9, take -16. . . . . Ans. 7.
35. From 12, take -8. . . . . Ans. 20
36. From -14, take -5. . . . . Ans. -9
37. From $3a-2b+6$, take $2a-7b-3$. . . . . Ans. $a+5b+9$.
38. From $13a-2b+9c-3d$, take $8a-6b+9c-10d+12$.
Ans. $5a+4b+7d-12$.
39. From $-7a+3m-8x$, take $-6a-5m-2x+3d$.
Ans. $-a+8m-6x-3d$.
40. From $32a+3b$, take $5a+17b$. . . . . . . Ans. $27a-14b$.
41. From $6a+5-3b$, take $-2a-9b-8$. . . Ans. $8a+6b+13$.
42. From $3c-2l+5c$, take $8l+7c-4l$. . . . . . Ans. $c-6l$.
43. From $3ax-2y^2$, take $-5ax-8y^2$. . . . . . Ans. $8ax+6y^2$.
44. From $2x^2-3a^2x^2+9$, take $x^2+5a^2x^2-3$. Ans. $x^2-8a^2x^2+12$.
45. From $4x^2y^3-5cz+8m$, take $-cz+2x^2y^3-4cz$.
Ans. $2x^2y^3+8m$.
46. From $x^3-11xyz+3a$, take $-6xyz+7-2a-5xyz$.
Ans. x^3+5a-7.
47. From $5(x+y)$, take $2(x+y)$. . . . . Ans. $3(x+y)$

48. From $3a(x-z)$, take $a(x-z)$. . . . . Ans. $2a(x-z)$.
49. From $7a^2(c-z)-ab(c-d)$, take $5a^2(c-z)-5ab(c-d)$.
Ans. $2a^2(c-z)+4ab(c-d)$.

ART. **59.** It is sometimes convenient to *indicate* the subtraction of a polynomial without actually performing the operation. This may be done, if it is a monomial, by placing the sign minus before it; and, if it is a polynomial, by enclosing it in a parenthesis, and then placing the sign minus before it.

Thus, to subtract $a-b$ from $2a$, we may write it $2a-(a-b)$, which reduces to $a+b$.

By this transformation, the same polynomial may be written in several different forms; thus:

$$a-b+c-d=a-b-(d-c)=a-d-(b-c)=a-(b-c+d).$$

Let the pupil, in each of the following examples, introduce all the quantities, except the first, into a parenthesis, and precede it by the sign minus, without altering the value of the expression.

1. $a-b+c$. . . . . Ans. $a-(b-c)$.
2. $b+c-d$. . . . . Ans. $b-(d-c)$.
3. $x^2-2xy+z$. . . . . Ans. $x^2-(2xy-z)$.
4. $ax+bc-cd+h$. . . . . Ans. $ax-(cd-bc-h)$.
5. $m-n-z-s$. . . . . Ans. $m-(n+z+s)$.
6. $m-n+z+s$. . . . . Ans. $m-(n-z-s)$.

It will be found a useful exercise for the pupil, to take each of the preceding polynomials, and without changing their values, write them in all possible modes, by including either two or more terms in a parenthesis.

OBSERVATIONS ON ADDITION AND SUBTRACTION.

ART. **60.** It has been shown, that Algebraic Addition is the process of collecting, into one, the quantities contained in two or more expressions. The pupil has already learned, that these expressions may be all positive, or all negative, or partly positive and partly negative. If they are either all positive, or all negative, the sum will be greater than either of the individual quantities; but, if some of the quantities are positive and others negative, the aggregate may be less than either of them, or, it may even be

REVIEW.—58. In subtracting $b-c$ from a, after taking away b, have we subtracted too much, or too little? What must be added, to obtain the true result? Why? What is the general rule for finding the difference between two algebraic quantities? 59. How can the subtraction of an algebraic quantity be indicated?

nothing. Thus, the sum of $+4a$ and $-3a$, is a; while that of $+a$ and $-a$, is zero, or 0.

As the pupil should have clear views of the use and meaning of the various expressions employed, it may be asked, what idea is he to attach to the operations of algebraic addition and subtraction.

ART. **61.** In common or arithmetical addition, when we say, that the sum of 5 and 3 is 8, we mean, that their sum is 8 greater than 0. In algebra, when we say that 5 and -3 are equal to 2, we mean, that the aggregate effect of adding 5 and subtracting 3, is the same as that of adding 2. When we say, that the sum of -5 and $+3$, is -2, we mean, that the result of subtracting 5, and adding 3, is the same as that of subtracting 2. Some algebraists say, that numbers with a positive sign represent quantities greater than 0, while those with a negative sign, such as -3, represent quantities *less than nothing.* The phrase, *less than nothing*, however, can not convey an intelligible idea, with any signification that would be attached to it in the ordinary use of language; but, if we are to understand by it, that any negative quantity, when added to a positive quantity, will produce a result *less than if nothing had been added to it;* or, that a negative quantity, when subtracted from a positive quantity, will produce a result *greater than if nothing had been taken from it,* then the phrase has a correct meaning. The idea, however, would be properly expressed, by saying, that negative quantities are *relatively* less than zero. Thus, if we take any number, for instance 10, and add to it the numbers 3, 2, 1, 0, -1, -2, and -3, we see, that adding a negative number produces a *less* result than adding zero.

10	10	10	10	10	10	10
3	2	1	0	-1	-2	-3
13	12	11	10	9	8	7

From this, we also see, that adding a negative number, produces the same result, as subtracting an equal positive number.

Again, if we take any number, for example 10, and subtract from it the numbers 3, 2, 1, 0, -1, -2, and -3, we see, that subtracting a negative number produces a *greater* result than subtracting zero:

10	10	10	10	10	10	10
3	2	1	0	-1	-2	-3
7	8	9	10	11	12	13

From this, we also see, that subtracting a negative number, produces the same result, as adding an equal positive number.

REVIEW.—60. When is the sum of two algebraic quantities less than either of them? When is the sum equal to zero?

ART. **62.** In consequence of the results they produce, it is customary to say, of two negative algebraic quantities, that the least is that which contains the *greatest* number of units. Thus, —3 is said to be less than —2. But, of two negative quantities, that which contains the greatest number of units is said to be *numerically* the greatest; thus, —3 is numerically greater than —2.

ART. **63.** A correct idea of the nature of the addition of positive and negative quantities, may be gained by the consideration of such questions as the following:

Suppose the sums of money put into a drawer to be positive quantities, and those taken out to be negative; how will the money in the drawer be affected, if, in one day, there are 20 dollars taken out, afterwards 15 dollars put in, after this 8 dollars taken out, and then 10 dollars put in? Or, in other words, what is the sum of —20, +15, —8, and +10? The answer, evidently, is —3; that is, the result of the whole operation diminishes the amount of money in the drawer 3 dollars. Had the sum of the quantities been positive, the result of the operation would have been, an increase of the amount of money in the drawer.

Again, suppose latitude north of the equator to be reckoned +, and that south, —; and that the degrees over which a ship sails north, are designated by +, while those she sails over south, are designated by —, and that we have the following question: A ship, in latitude 10 degrees north, sails 5 degrees south, then 7 degrees north, then 9 degrees south, and then 3 degrees north; what is her present latitude?

This question is the same as to find the sum of the quantities +10, —5, +7, —9, and +3; this is evidently +6; that is, the ship is in 6 degrees north latitude. Had the sum of the negative numbers been the greater, it follows, that the ship would have been found in south latitude.

Other questions of a similar nature may be used by the instructor, to illustrate the subject.

ART. **64.** Subtraction, in arithmetic, shows the method of finding the excess of one quantity over another of the same kind. In this case, the number to be subtracted must be less than that from which it is to be taken; and, as they are considered without refer-

REVIEW.—61. What is meant, by saying that the sum of +5 and —3, is equal to +2? What is meant, by saying that the sum of —5 and +3, is equal to —2? Is it correct to say, that any quantity is less than nothing? What is the effect of adding a positive quantity? Of adding a negative quantity? Of subtracting a positive quantity? Of subtracting a negative quantity? 62. In comparing two negative algebraic quantities, which is called the least? Which is numerically the greatest?

ence to sign, it is equivalent to regarding them of the same sign. Algebraic Subtraction shows the method of finding the *difference* between two quantities which have either the same or unlike signs; and it frequently happens, that this difference is greater than either of the quantities. To understand this properly, requires a knowledge of the nature of positive and negative quantities.

All quantities are to be regarded as positive, unless, for some special reason, they are otherwise designated. Negative quantities embrace those that are, in their nature, the *opposite* of positive quantities.

Thus, if a merchant's gains are positive, his losses are negative; if latitude north of the equator is reckoned $+$, that south, would be $-$; if distance to the right of a certain line is reckoned $+$, then distance to the left would be $-$; if elevation above a certain point, or plane, is regarded as $+$, then distance below would be $-$; if time after a certain hour is $+$, then time before that hour is $-$; if motion in one direction is $+$, then motion in an opposite direction would be $-$; and so on.

With this knowledge of the meaning of the sign minus, it is easy to see how the difference of two quantities having the same sign, is equal to their difference; and also, how the difference of two quantities having different signs, is equal to their sum.

1. One place is situated 10, and another 6 degrees north of the equator, what is their difference of latitude?

Here we are required to find the difference between $+10$ and $+6$, which is evidently $+4$; by which we are to understand that the first place is 4 degrees farther north than the second.

2. Two places are situated, one in 10, and the other in 6 degrees south latitude; what is the difference of latitude?

Here we are required to find the difference between -6 and -10, which is evidently -4, by which we learn, that the first place is 4 degrees farther south than the second.

3. One place is situated in 10 degrees north, and another in 6 degrees south latitude; what is their difference of latitude?

Here we are required to find the difference between $+10$ and -6, or to take -6 from $+10$, which, by the rule for subtraction, leaves $+16$; which is evidently the difference of their latitudes, and from which we learn, that the first place is 16 degrees farther north than the other.

It is thus, when properly understood, the results are always capable of a satisfactory explanation.

Review.—64. In what respects does algebraic differ from arithmetical Subtraction? In what respect do negative quantities differ from positive? Illustrate the difference by examples.

MULTIPLICATION.

ART. **65.** MULTIPLICATION, in Algebra, is the process of taking one algebraic expression, as often as there are units in another.

In Algebra, as in Arithmetic, the quantity to be multiplied is called the *multiplicand;* the quantity by which we multiply, the *multiplier*, and the result of the operation, the *product.* The multiplicand and multiplier are generally called *factors.*

ART. **66.** Since the quantity a, taken once, is represented by a, when taken twice, by $a+a$, or $2a$, when taken three times, by $a+a+a$, or $3a$, it is evident, that *to multiply a literal quantity by a number, it is only necessary to write the multiplier as the coëfficient of the literal quantity.*

1. If 1 lemon costs a cents, how many cents will 5 lemons cost?

If *one* lemon costs a cents, *five* lemons will cost *five* times as much, that is $5a$ cents.

2. If 1 orange costs c cents, how many cents will 6 oranges cost?

3. A merchant bought a pieces of cloth, each containing b yards, at c dollars per yard; how many dollars did the whole cost?

In a pieces, the number of yards would be represented by ab, or ba, and the cost of ab yards at c dollars per yard, would be represented by c taken ab times, that is, by $ab\times c$, which is represented by abc.

ART. **67.** It is shown in "Ray's Arithmetic," Part III, Art. 30, that the product of two factors is the same, whichever be made the multiplier; we will, however, demonstrate the principle here.

Suppose we have a sash containing a vertical, and b horizontal rows; there will be a panes in each horizontal row, and b panes in each vertical row; it is required to find the number of panes in the window.

It is evident, that the whole number of panes in the window will be equal to the number in one row, taken as many times as there are rows. Then, since there are a vertical rows, and b panes in each row, the whole number of panes will be represented by b taken a times, that is, by ab.

Again, since there are b horizontal rows, and a panes in each row, the whole number of panes will be represented by a taken b times, that is, by ba. But, since either of the expressions, ba or

REVIEW.—65. What is Multiplication in Algebra? What is the multiplicand? The multiplier? The product? What are the multiplicand and multiplier generally called? 66. How do you multiply a literal quantity by a number?

ab, represents the whole number of panes in the window, they are equal to each other, that is, ab is equal to ba. Hence, it follows, that *the product of two factors is the same, whichever be made the multiplier.*

By taking $a=3$ and $b=4$, the figure in the margin may be used to illustrate the principle in a particular case.

In the same manner, the product of three or more quantities is the same, in whatever order they are taken. Thus, $2\times3\times4=3\times2\times4=4\times2\times3$, since the product in each case is 24.

1. What will 2 boxes, each containing a lemons, cost at b cents per lemon?

One box will cost ab cents, and 2 boxes will cost twice as much as 1 box, that is, $2ab$ cents.

2. What is the product of $2b$, multiplied by $3a$?

The product will be represented by $2b\times3a$, or by $3a\times2b$, or by $2\times3\times ab$, since the product is the same, in whatever order the factors are placed. But 2×3 is equal to 6, hence the product $2b\times3a$ is equal to $6ab$.

Hence, we see, that in multiplying one monomial by another, *the coëfficient of the product is obtained by multiplying together the coëfficients of the multiplicand and multiplier.* This is termed, *the rule of the coëfficients.*

Art. **68.** Since the product of two or more factors is the same, in whatever order they are written, if we take the product of any two factors, as 2×3, and multiply it by any number, as 5, the product may be written $5\times2\times3$, or $5\times3\times2$, that is, 10×3, or 15×2, either of which is equal to 30. From which we see, that *when either of the factors of a product is multiplied, the product itself is multiplied.*

Art. **69.**—1. What is the product of a by a?

The product of b by a is written ab, hence, the product of a by a would be written aa; but this, (Art. 33,) for the sake of brevity, is written a^2.

2. What is the product of a^2 by a?

Since a^2 may be written thus, aa, the product of a^2 by a may be

Review.—67. Prove that 3 times 4 is the same as 4 times 3. Prove that a times b is the same as b times a. Is the product of any number of factors changed by altering their arrangement? In multiplying one monomial by another, how is the coëfficient of the product obtained? 68. If you multiply one of the factors of a product, how does it affect the product? 69. How may the product of a by a be written? How may the product of a^2 by a be written?

expressed thus, $aa\times a$, or aaa, which, for the sake of brevity, is written a^3. Hence, *the exponent of a letter in the product, is equal to the sum of its exponents in the two factors.* This is termed, *the rule of the exponents.*

3. What is the product of a^2 by a^2? . . . . Ans. $aaaa$, or a^4.
4. What is the product of a^2b by ab? . . Ans. $aaabb$, or a^3b^2.
5. What is the product of $2ab^2$ by $3ab$? Ans. $6aabbb$, or $6a^2b^3$.

Hence, the

RULE,

FOR MULTIPLYING ONE POSITIVE MONOMIAL BY ANOTHER.

Multiply the coëfficients of the two terms together, and to their product annex all the letters in both quantities, giving to each letter an exponent equal to the sum of its exponents in the two factors.

NOTE.—It is customary to write the letters in the order of the alphabet. Thus, $ab\times c$ is generally written abc.

6. Multiply ab by x. . . . . Ans. abx.
7. Multiply $2bc$ by mn. . . . . Ans. $2bcmn$.
8. Multiply $4ab$ by $5xy$. . . . . Ans. $20abxy$.
9. Multiply $7ax$ by $4cd$. . . . . Ans. $28acdx$.
10. Multiply $6by$ by $3ax$. . . . . Ans. $18abxy$.
11. Multiply $3a^2b$ by $4ab$. . . . . Ans. $12a^3b^2$.
12. Multiply $2xy^2$ by $3x^2y$. . . . . Ans. $6x^3y^3$.
13. Multiply $4ab^2x$ by $5ax^2y$. . . . . Ans. $20a^2b^2x^3y$.
14. What is the product of $3a^3b^2c$ by $5ab^2c^3$? . . Ans. $15a^4b^4c^4$.
15. What is the product of $7xy^2z$ by $8x^3yz$? . . Ans. $56x^4y^3z^2$.

NOTE.—The learner must be careful to distinguish between the coëfficient and the exponent. Thus, $2a$ is different from a^2. To fix this in his mind, let him answer such questions as the following:

What is $2a-a^2$ equal to, when a is 1? . . . . Ans. 1.
What is a^2-2a equal to, when a is 5? . . . . Ans. 15.
What is a^3-3a equal to, when a is 4? . . . . Ans. 52.
What is a^4-4a equal to, when a is 3? . . . . Ans. 69.

ART. **70.**—1. Suppose you purchase 5 oranges at 4 cents a piece, and pay for them, and then purchase 2 lemons at the same price; what will be the cost of the whole?

5 oranges, at 4 cents each, will cost 20 cents; 2 lemons, at 4 cents each, will cost 8 cents, and the cost of the whole will be $20+8=28$ cents.

The work may be written thus:

$$\begin{array}{r} 5+2 \\ 4 \\ \hline 20+8=28 \text{ cents.} \end{array}$$

If you purchase a oranges at c cents a piece, and b lemons at c cents a piece, what will be the cost of the whole?

The cost of a oranges, at c cents each, will be ac cents; the cost of b lemons, at c cents each, will be bc cents, and the whole cost will be $ac+bc$ cents.

The work may be written thus:

$$\begin{array}{r} a+b \\ c \\ \hline ac+bc \end{array}$$

Hence, when the sign of each term is positive, we have the following

RULE,

FOR MULTIPLYING A POLYNOMIAL BY A MONOMIAL.

Multiply each term of the multiplicand by the multiplier.

EXAMPLES.

2. Multiply $a+d$ by b. . . . . Ans. $ab+bd$.
3. Multiply $ac+bc$ by d. . . . . Ans. $acd+bcd$.
4. Multiply $4x+5y$ by $3a$. . . . . Ans. $12ax+15ay$.
5. Multiply $2x+3z$ by $2b$. . . . . Ans. $4bx+6bz$.
6. Multiply $m+2n$ by $3n$. . . . . Ans. $3mn+6n^2$.
7. Multiply $x+y$ by ax. . . . . Ans. ax^2+axy.
8. Multiply x^2+y^2 by xy. . . . . Ans. x^3y+xy^3.
9. Multiply $2x+5y$ by abx. . . . . . . Ans. $2abx^2+5abxy$.
10. Multiply $3x^2+2xz$ by $2xz$. . . . . Ans. $6x^3z+4x^2z^2$.
11. Multiply $3a+2b+5c$ by $4d$. . . . Ans. $12ad+8bd+20cd$.
12. Multiply $bc+af+mx$ by $3ax$. . Ans. $3abcx+3a^2fx+3amx^2$.
13. Multiply $ab+ax+xy$ by $abxy$. . Ans. $a^2b^2xy+a^2bx^2y+abx^2y^2$.

ART. 71.—1. Let it be required to find the product of $x+y$ by $a+b$. Here the multiplicand is to be taken as many times as there are units in $a+b$, and the whole product will evidently be equal to the sum of the two partial products. Thus,

$$\begin{array}{ll} x+y & \\ a+b & \\ \hline ax+ay & =\text{the multiplicand taken } a \text{ times.} \\ \quad\quad bx+by & =\text{the multiplicand taken } b \text{ times.} \\ \hline ax+ay+bx+by & =\text{the multiplicand taken } (a+b) \text{ times.} \end{array}$$

If $x=5$, $y=6$, $a=2$, and $b=3$, the multiplication may be arranged thus:

$$\begin{array}{ll} 5+6 & \\ 2+3 & \\ \hline 10+12 & =\text{the multiplicand taken 2 times.} \\ \quad\quad 15+18 & =\text{the multiplicand taken 3 times.} \\ \hline 10+27+18=55 & =\text{the multiplicand taken 5 times.} \end{array}$$

Hence, when all the terms in each are positive, we have the following

RULE,

FOR MULTIPLYING ONE POLYNOMIAL BY ANOTHER.

Multiply each term of the multiplicand by each term of the multiplier, and add the products together.

2.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

3.

$$\begin{array}{r} a^2b+cd \\ ab+cd^2 \\ \hline a^3b^2+abcd \\ +a^2bcd^2+c^2d^3 \\ \hline a^3b^2+a^2bcd^2+abcd+c^2d^3 \end{array}$$

4. Multiply $a+b$ by $c+d$. . . . . . . Ans. $ac+ad+bc+bd$.
5. Multiply $2x+3y$ by $3a+2b$. . . Ans. $6ax+9ay+4bx+6by$.
6. Multiply $2a+3b$ by $3c+d$. . . Ans. $6ac+9bc+2ad+3bd$.
7. Muliply $m+n$ by $x+z$. . . . . Ans. $mx+nx+mz+nz$.
8. Multiply $4a+3b$ by $2a+b$. . . . . Ans. $8a^2+10ab+3b^2$.
9. Multiply $4x+5y$ by $2a+3x$. Ans. $8ax+10ay+12x^2+15xy$.
10. Multiply $3x+2y$ by $2x+3y$. . . . . Ans. $6x^2+13xy+6y^2$.
11. Multiply a^2+b^2 by $a+b$, ... Ans. $a^3+a^2b+ab^2+b^3$.
12. Multiply $3a^2+2b^2$ by $2a^2+3b^2$. . . Ans. $6a^4+13a^2b^2+6b^4$.
13. Multiply a^2+ab+b^2 by $a+b$. . . Ans. $a^3+2a^2b+2ab^2+b^3$.
14. Multiply c^3+d^3 by $c+d$. . . . . Ans. $c^4+cd^3+c^3d+d^4$.
15. Multiply $x^2+2xy+y^2$ by $x+y$. . . Ans. $x^3+3x^2y+3xy^2+y^3$.

SIGNS.

ART. **72.** In the preceding examples, it was assumed that the product of two positive quantities, is also positive. It may, however, be shown as follows:

1st. Let it be required to find the product of $+b$ by a.

The quantity b, taken once, is $+b$; taken twice, is evidently, $+2b$; taken 3 times, is $+3b$, and so on. Therefore, taken a times, it is $+ab$. Hence, the product of two positive quantities is positive; or, as it may be more briefly expressed, *plus* multiplied by *plus*, gives *plus*.

2d. Let it be required to find the product of $-b$ by a.

REVIEW.—To what is the exponent of a letter in the product equal? What is the rule for multiplying one positive monomial by another! 70. What is the product of a plus b, by c? When all the terms in each are positive, how do you multiply a polynomial by a monomial? 71. When all the terms in each are positive, how do you find the product of two polynomials?

The quantity $-b$, taken once, is $-b$; taken twice, is $-2b$; taken 3 times, is $-3b$; and hence, taken a times, is $-ab$; that is, a *negative* quantity multiplied by a *positive* quantity, gives a *negative* product. This is generally expressed, by saying, that *minus* multiplied by *plus*, gives *minus*.

3d. Let it be required to multiply b by $-a$.

Since, when two quantities are to be multiplied together, either may be made the multiplier (Art. 67), this is the same as to multiply $-a$ by b, which gives $-ab$. That is, a positive quantity multiplied by a negative quantity, gives a negative product; or, more briefly, *plus* multiplied by *minus*, gives *minus*.

4th. Let it be required to multiply -3 by -2.

The negative multiplier signifies, that the multiplicand is to be taken positively, as many times as there are units in the multiplier, and then subtracted. The product of -3 by $+2$ is -6, then, changing the sign to subtract, the -6 becomes $+6$; and, in the same manner, the product of $-b$ by $-a$ is $+ab$.

Hence, the product of two negative quantities is positive; or, more briefly, *minus* multiplied by *minus*, gives *plus*.

NOTE.—The following proof of the last principle, that the product of two negative quantities is positive, is generally regarded by mathematicians as more satisfactory than the preceding, though it is not quite so simple. The instructor can use either method.

5th. To find the product of two negative quantities.

To do this, let us find the product of $c-d$ by $a-b$.

Here it is required to take $c-d$ as many times as there are units in $a-b$. It is obvious that this will be done by taking $c-d$ as many times as there are units in a, and then subtracting from this product, $c-d$ taken as many times as there are units in b.

Since plus multiplied by plus gives plus, and minus multiplied by plus gives minus, the product of $c-d$ by a, is $ac-ad$.

In the same manner, the product of $c-d$ by b, is $bc-bd$; changing the signs of the last product to subtract it, it becomes $-bc+bd$; hence the product of $c-d$ by $a-b$, is $ac-ad-bc+bd$.

But the last term, $+bd$, is the product of $-d$ by $-b$, hence the product of two negative quantities is positive; or, more briefly, *minus* multiplied by *minus* produces *plus*.

The multiplication of $c-d$ by $a-b$ may be written thus:

$$\begin{array}{ll}
c-d & \\
\underline{a-b} & \\
ac-ad=c-d & \text{taken } a \text{ times.} \\
\underline{\quad -bc+bd=c-d} & \text{taken } b \text{ times, and then subtracted.} \\
ac-ad-bc+bd &
\end{array}$$

The operation may be illustrated by figures; thus, let it be required to find the product of 7—4 by 5—3.

7—4 5—3 35—20 —21+12 35—41+12	We first take 5 times 7—4; this gives a product too great, by 3 times 7—4, or 21—12, which, being subtracted from the first product, gives for the true result, 35—41+12, which reduces to +6. This is evidently correct, for 7—4 =3, and 5—3=2, and the product of 3 by 2 is 6.

From the preceding illustrations, we derive the following

GENERAL RULE,

FOR THE SIGNS.

Plus multiplied by plus, or minus multiplied by minus, gives plus.
Plus multiplied by minus, or minus multiplied by plus, gives minus.
Or, *the product of like signs gives plus, and of unlike signs gives minus*

From all the preceding, we derive the

GENERAL RULE,

FOR THE MULTIPLICATION OF ALGEBRAIC QUANTITIES.

Multiply every term of the multiplicand, by each term of the multiplier; observing,

1st. *That the coëfficient of any term is equal to the product of the coëfficients of its factors.*

2d. *That the exponent of any letter in the product is equal to the sum of its exponents in the two factors.*

3d. *That the product of like signs, gives plus in the product, and unlike signs, gives minus. Then, add the several partial products together.*

NUMERICAL EXAMPLES,

TO VERIFY THE RULE OF THE SIGNS.

1. Multiply 8—3 by 5. . . . . Ans. $40-15=25=5\times5$.
2. Multiply 20—13 by 4. . . . . . . Ans. $80-52=28=7\times4$.
3. Multiply 13—7 by 11—8 . Ans. $143-181+56=18=6\times3$.
4. Multiply 10+3 by 3—5.
   Ans. $30-41-15=-26=13\times-2$.
5. Multiply 9—5 by 8—2. . . Ans. $72-58+10=24=4\times6$.
6. Multiply 8—7 by 5—3. . . . Ans. $40-59+21=2=1\times2$.

REVIEW.—72. What is the product of $+b$ by $+a$? Why? What is the product of $-b$ by a? Why? What is the product of $+b$ by $-a$? Why? What is the product of —3 by —2? What does a negative multiplier signify? What does minus multiplied by minus produce? What is the general rule for the signs? What is the general rule for the multiplication of algebraic quantities?

GENERAL EXAMPLES.

1. Multiply $3a^2xy$ by $7axy^3$. . . . . Ans. $21a^3x^2y^4$.
2. Multiply $-5a^2b$ by $3ab^3$. . . . . Ans. $-15a^3b^4$.
3. Multiply $-5x^2y$ by $-5xy^2$. . . . . Ans. $25x^3y^3$
4. Multiply $3a-2b$ by $4c$. . . . . Ans. $12ac-8bc$.
5. Multiply $3x+2y$ by $-2x$. . . . . Ans. $-6x^2-4xy$.
6. Multiply $a+b$ by $x-y$. . . . . Ans. $ax-ay+bx-by$.
7. Multiply $a-b$ by $a-b$. . . . . Ans. $a^2-2ab+b^2$.
8. Multiply a^2+ac+c^2 by $a-c$. . . . . Ans. a^3-c^3.
9. Multiply $m+n$ by $m-n$. . . . . Ans. m^2-n^2.
10. Multiply $a^2-2ab+b^2$ by $a+b$. . . . Ans. $a^3-a^2b-ab^2+b^3$.
11. Multiply $3x^3y-2xy^3+y^4$ by $2xy+y^2$.
Ans. $6x^4y^2+3x^3y^3-4x^2y^4+y^6$.
12. Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$. . Ans. $a^4-2a^2b^2+b^4$.
13. Multiply y^2-y+1 by $y+1$. . . . . Ans. y^3+1.
14. Multiply x^2+y^2 by x^2-y^2. . . . . Ans. x^4-y^4.
15. Multiply a^2-3a+8 by $a+3$. . . . . . Ans. a^3-a+24.
16. Multiply $2x^2-3xy+y^2$ by x^2-5xy.
Ans. $2x^4-13x^3y+16x^2y^2-5xy^3$.
17. Multiply $3a+5b$ by $3a-5b$. . . . . . . Ans. $9a^2-25b^2$.
18. Multiply $2a^2-4ax+2x^2$ by $3a-3x$.
Ans. $6a^3-18a^2x+18ax^2-6x^3$.
19. Multiply $5x^3+3y^3$ by $5x^3-3y^3$. . . . . . Ans. $25x^6-9y^6$.
20. Multiply $2a^3+2a^2x+2ax^2+2x^3$ by $3a-3x$. . Ans. $6a^4-6x^4$.
21. Multiply $3a^2+3ax+3x^2$ by $2a^2-2ax$. . . Ans. $6a^4-6ax^3$.
22. Multiply $3a^2+5ax-2x^2$ by $2a-x$.
Ans. $6a^3+7a^2x-9ax^2+2x^3$.
23. Multiply $x^6+x^4+x^2$ by x^2-1. . . . . Ans. x^8-x^2.
24. Multiply x^2+xy+y^2 by x^2-xy+y^2. . . Ans. $x^4+x^2y^2+y^4$.
25. Multiply $a^3+a^2b+ab^2+b^3$ by $a-b$. . . . . Ans. a^4-b^4.

In the following examples, let the pupil perform the multiplications indicated, by multiplying together the quantities contained in the parentheses.

26. $(x-3)(x-3)(x-3)$. . . . . Ans. $x^3-9x^2+27x-27$.
27. $(x-4)(x-5)(x+4)(x+5)$. . . . . . Ans. x^4-41x^2+400.
28. $(a+c)(a-c)(a+c)(a-c)$. . . . . Ans. $a^4-2a^2c^2+c^4$.
29. $(a^2+b^2+c^2-ab-ac-bc)(a+b+c)$. . Ans. $a^3+b^3+c^3-3abc$.
30. $(n^2+n+1)(n^2+n+1)(n-1)(n-1)$. . . Ans. n^6-2n^3+1.

DIVISION.

ART. **73.** DIVISION in Algebra, is the process of finding how often one algebraic quantity is contained in another.

Or, it may be defined thus: Having the product of two factors, and one of them given, Division teaches the method of finding the other.

The number by which we divide, is called the *divisor;* the number to be divided, is called the *dividend;* the number of times the divisor is contained in the dividend, is called the *quotient.*

ART. **74.** Since Division is the reverse of Multiplication, the quotient, multiplied by the divisor, must produce the dividend.

The usual method of indicating Division, is to write the divisor under the dividend in the form of a fraction. Thus, to indicate that ab is to be divided by a, we write, $\frac{ab}{a}$. Algebraic Division, however, is sometimes indicated, like that of whole numbers, thus, $a)ab$; where a is the divisor, and ab the dividend.

NOTE TO TEACHERS.—In solving the following examples, let the pupil give the reason for the answer, as in the solution to the first question. Although the examples can be solved mentally, it will be found most advantageous, to work them on the slate, or blackboard; as the learner, by this means, will be preparing for the performance of more difficult operations.

1. How often is x contained in $4x$? . . . . . . Ans. $\frac{4x}{x}=4$.

This solution is to be given by the pupil, thus: $4x$ divided by x, is equal to 4, because the product of 4 by x is $4x$.

2. How often is a contained in $6a$? . . . . Ans. 6.
3. How often is a contained in ab? . . . . Ans. b.
4. How often is b contained in $3ab$? . . . . . . Ans. $3a$.
5. How often is a contained in abx? . . . . . . Ans. bx.
6. How often is a contained in $5abx$? . . . . . . Ans. $5bx$.
7. How often is 2 contained in $4a$? . . . . Ans. $2a$.
8. How often is $2a$ contained in $4ab$? . . . . . . Ans. $2b$.
9. How often is a contained in a^2? . . . . Ans. a.
10. How often is a contained in a^3? . . . . Ans. a^2.
11. How often is a contained in $3a^2$? . . . . . . Ans. $3a$.
12. How often is ab contained in $5a^2b$? . . . . . Ans. $5a$.

REVIEW.—73. What is Algebraic Division? What is the divisor? The dividend? The quotient? 74. To what is the product of the quotient and the divisor equal? Why? What is the usual method of indicating division?

13. How often is $2a$ contained in $10a^3$? . . . . . Ans. $5a^2$
14. How often is $3a^2$ contained in $12a^3b$? . . . . . Ans. $4ab$.
15. How often is $4ab^2$ contained in $12a^3b^3c$? . . Ans. $3a^2bc$.
16. How often is $2a^2$ contained in $6a^5b$?

$$\text{Solution, } \frac{6a^5b}{2a^2}=\frac{6}{2}a^{5-2}b=3a^3b.$$

In obtaining this quotient, we readily see,

1st. The coëfficient of the quotient, must be such a number, that when multiplied by 2, it shall produce 6; hence, to obtain it, we divide 6 by 2.

2d. The exponent of a must be such a number, that when 2, the exponent of a in the divisor, is added to it, the sum shall be 5; hence, to obtain it, we must subtract 2 from 5; that is, 5—2 is equal to 3, the exponent of a in the quotient.

3d. The letter b, which is a factor of the dividend, but not of the divisor, must be found in the quotient, in order that the product of the divisor and quotient may equal the dividend.

ART. **75.** It remains to ascertain the rule for the signs.

Since $+a$ multiplied by $+b=+ab$, therefore, $\frac{+ab}{+b}=+a$; hence, *plus* divided by *plus*, gives *plus*.

Since $-a$ multiplied by $+b=-ab$, therefore, $\frac{-ab}{+b}=-a$; hence, *minus* divided by *plus*, gives *minus*.

Since $+a$ multiplied by $-b=-ab$, therefore, $\frac{-ab}{-b}=+a$; hence, *minus* divided by *minus*, gives *plus*.

Since $-a$ multiplied by $-b=+ab$, therefore, $\frac{+ab}{-b}=-a$; hence, *plus* divided by *minus*, gives *minus*.

From this, we see, that in Division, *like* signs give *plus*, and *unlike* signs give *minus*.

Hence the

RULE

FOR DIVIDING ONE MONOMIAL BY ANOTHER.

Divide the coëfficient of the dividend, by that of the divisor; observing, that like signs give plus, and unlike signs give minus.

After the coëfficient, write the letters common to both divisor and dividend, giving to each an exponent, equal to the excess of the exponent of the same letter in the dividend, over that in the divisor.

In the quotient, write the letters with their respective exponents, that are found in the dividend, but not in the divisor.

NOTE.—The pupil must recollect, that when a letter has no exponent expressed, 1 is understood; thus, a, is the same as a^1.

EXAMPLES.

17. Divide $15a^3bc$ by $3a^2b$. . . . . Ans. $5ac$.
18. Divide $27x^2y^2$ by $-3xy$. . . . . Ans. $-9xy$.
19. Divide $-18a^3x$ by $-6ax$. . . . . Ans. $3a^2$.
20. Divide $25abxy$ by $5ay$. . . . . Ans. $5bx$.
21. Divide a^4x^3 by a^2x. . . . . Ans. a^2x^2.
22. Divide $-4a^5x^2$ by $2a^3x$. . . . . Ans. $-2a^2x$.
23. Divide $-12c^4x^3y^5$ by $-4c^4xy^2$. . . . . Ans. $3x^2y^3$.
24. Divide $-24a^4x^3y^2v$ by $4a^2xyv$. . . . . . . Ans. $-6a^2x^2y$.
25. Divide $6acx^2y^6v$ by $3ax^2y^4v$. . . . . Ans. $2cy^2$.
26. Divide $-10c^2x^5y^5v$ by $-2cy^4v$. . . . . Ans. $5cx^5y$.
27. Divide $60a^7x^2y^7$ by $12a^5y^3$. . . . . Ans. $5a^2x^2y^4$.
28. Divide $-18a^4c^8x^2v^9$ by $-6a^4c^2v^5$. . . . . . . Ans. $3c^6x^2v^4$.
29. Divide $-28ac^2x^8y^4v^2$ by $14ax^5y^4$. . . . . . Ans. $-2c^2x^3v^2$.
30. Divide $30ac^4e^4x^4y^2$ by $-2aex^4$ . . . . . Ans. $-15c^4e^3y^2$.

NOTE.—Although the method of operation in each of the following examples is the same as in the preceding, they may be passed over, until the book is reviewed.

31. Divide $(x+y)^2$ by $(x+y)$. . . . . Ans. $(x+y)$.
32. Divide $(a+b)^3$ by $(a+b)^2$. . . . . Ans. $(a+b)$.
33. Divide $(a+b)^4$ by $(a+b)$. . . . . Ans. $(a+b)^3$.
34. Divide $6(m+n)^3$ by $2(m+n)$. . . . . Ans. $3(m+n)^2$.
35. Divide $a^2(b+c)^2$ by $a(b+c)$. . . . . Ans. $a(b+c)$.
36. Divide $6a^2b(x+y)^3$ by $2ab(x+y)^2$. . . . . Ans. $3a(x+y)$.
37. Divide $(x+y)(a-b)^3$ by $(a-b)$. . . . Ans. $(x+y)(a-b)^2$.
38. Divide $(x-y)^3(m-n)^2$ by $(x-y)^2(m-n)^2$. . . Ans. $(x-y)$.

ART. **76.** It is evident, that one monomial cannot be divided by another, in the following cases.

1st. When the coëfficient of the dividend is not exactly divisible by the coëfficient of the divisor.

2d. When the same literal factor has a greater exponent in the divisor than in the dividend.

3d. When the divisor contains one or more literal factors, not found in the dividend.

In each of these cases, the division is to be indicated by writing the divisor under the dividend, in the form of a fraction. The

REVIEW.—75. When the signs of the dividend and divisor are alike, what will be the sign of the quotient? Why? When the signs of the dividend and divisor are unlike, what will be the sign of the quotient? Why? What is the rule for dividing one monomial by another?

fraction thus found, may often be reduced to lower terms. For the method of doing this, see Art. 129.

ART. 77. It has been shown, in Art. 68, that any product is multiplied, by multiplying either of its factors; hence, conversely, *any dividend will be divided, by dividing either of its factors.*

Thus, $\frac{4\times6}{2}=2\times6=12$, by dividing the factor 4.

Or, $\frac{4\times6}{2}=4\times3=12$, by dividing the factor 6.

DIVISION OF A POLYNOMIAL BY A MONOMIAL.

ART. 78. Since, in multiplying a polynomial by a monomial, we multiply each term of the multiplicand by the multiplier; therefore, we have the following

RULE,

FOR DIVIDING A POLYNOMIAL BY A MONOMIAL.

Divide each term of the dividend, by the divisor, according to the rule for the division of monomials.

EXAMPLES.

1. Divide $6x+12y$ by 3. . . . . Ans. $2x+4y$.
2. Divide $15x-20b$ by 5. . . . . Ans. $3x-4b$.
3. Divide $21a+35b$ by -7. . . . . Ans. $-3a-5b$.
4. Divide $6ax+9ay$ by $3a$. . . . . Ans. $2x+3y$.
5. Divide $ab+ac$ by a. . . . . Ans. $b+c$.
6. Divide $abc-acf$ by ac. . . . . Ans. $b-f$.
7. Divide $12ay-8ac$ by $-4a$. . . . . . . Ans. $-3y+2c$.
8. Divide $10ax-15ay$ by $-5a$. . . . . . . Ans. $-2x+3y$.
9. Divide $12bx-18x^2$ by $6x$. . . . . Ans. $2b-3x$.
10. Divide $a^2b^2-2ab^3x$ by ab. . . . . Ans. $ab-2b^2x$.
11. Divide $12a^2bc-9acx^2+6ab^2c$ by $-3ac$.
    Ans. $-4ab+3x^2-2b^2$.
12. Divide $15a^5b^2c-21a^2b^3c^2$ by $3a^2bc$. . . . Ans. $5a^3b-7b^2c$.
13. Divide $6a^3bc+2a^2bc^2-4a^2c^3$ by $2a^2c$. . Ans. $3ab+bc-2c^2$.

NOTE.—The following examples may be omitted until the book is reviewed.

14. Divide $6(a+c)+9(a+x)$ by 3. . . Ans. $2(a+c)+3(a+x)$.
15. Divide $5a(x+y)-10a^2(x-y)$ by $5a$. Ans. $(x+y)-2a(x-y)$.
16. Divide $a^2b(c+d)+ab^2(c^2-d)$ by ab. Ans. $a(c+d)+b(c^2-d)$.

REVIEW.—76. In what cases is the exact division of one monomial by another impossible? 78. What is the rule for dividing a polynomial by a monomial?

17. Divide $ac(m+n)-bc(m+n)$ by $m+n$. . . . . Ans. $ac-bc$.

18. Divide $12(a-b)^2+6c(a-b)^3$ by $2(a-b)$.
Ans. $6(a-b)+3c(a-b)^2$.

19. Divide $2a^2c(x+y)^3+2ac^2(x+y)^4$ by $2ac(x+y)^2$.
Ans. $a(x+y)+c(x+y)^2$.

20. Divide $(m+n)(x+y)^2+(m+n)(x-y)^2$ by $m+n$.
Ans. $(x+y)^2+(x-y)^2$.

DIVISION OF ONE POLYNOMIAL BY ANOTHER.

ART. **79.** To explain the method of dividing one polynomial by another, we may regard the dividend as a product, of which the divisor and the quotient are the two factors. We shall examine the method of forming this product, and then, by a reverse operation, explain the process of division.

Multiplication, or formation of a product.

$$\begin{array}{r}
2a^2-ab \\
\underline{a-b} \\
2a^3-a^2b \qquad\quad \\
\underline{-2a^2b+ab^2} \\
2a^3-3a^2b+ab^2
\end{array}$$

Division, or decomposition of a product.

$$\begin{array}{lr|l}
 & 2a^3-3a^2b+ab^2 & a-b \\ \cline{3-3}
 & \underline{2a^3-2a^2b} \qquad\quad & 2a^2-ab \\
\text{1st. rem.} & -a^2b+ab^2 & \\
 & \underline{-a^2b+ab^2} & \\
\text{2d. rem.} & 0 &
\end{array}$$

If we multiply $2a^2-ab$ by $a-b$, and arrange the terms according to the powers of a, we shall find the product to be $2a^3-3a^2b+ab^2$.

In this multiplication we remark,

1st. Since each term in the multiplicand is multiplied by each term in the multiplier, if no reduction takes place in adding the several partial products together, the *number* of terms in the final product will be equal to the number produced by multiplying together the number of terms in the two factors. Thus, if one factor have 3 terms, and the other 2, the number of terms in the product will be six. Frequently, however, a reduction takes place, by which the number of terms is lessened. Thus, in the above example, two terms being added together, there are only 3 terms in the product.

2d. In every case of multiplication, there are two terms which can never be united with any other. These are, first: that term which is the product of the two terms in the factors, which contain the *highest* power of the same letter; and second: the term which is the product of the two terms in the factors, which contain the *lowest* power of the same letter.

From the last principle it follows, that if the term containing the highest power of any letter in the dividend, be divided by the term containing the highest power of the same letter in the divisor,

the result will be the term of the quotient containing the highest power of that letter. Hence, if $2a^3$ be divided by a, the result, $2a^2$, will be the term containing the highest power of a, in the quotient.

The dividend expresses the sum of the partial products of the divisor, by the different terms of the quotient. If, then, we form the product of the divisor by the first term, $2a^2$, of the quotient, and subtract it from the dividend, the remainder, $-a^2b+ab^2$, will be the sum of the other partial products of the divisor, by the remaining terms of the quotient.

Now, since this remainder is produced, by multiplying the divisor by the remaining terms of the quotient, it follows, as in the method of obtaining the first term of the quotient, that if the term containing the *highest* power of a particular letter in this remainder, be divided by the term containing the highest power of the same letter in the divisor, the quotient will be the term containing the highest power of that letter in the remaining terms of the quotient.

Hence, if $-a^2b$ be divided by a, the quotient, $-ab$, will be another term of the quotient. Multiplying the divisor by this second term, and subtracting, we find the second remainder is 0; hence, the exact quotient is $2a^2-ab$. Had there been a second remainder, the third term of the quotient would have been obtained from it in the same manner as the second term was obtained from the first remainder.

Since each term of the quotient is found, by dividing that term of the dividend containing the highest power of a particular letter, by the term of the divisor containing the highest power of the same letter, it is more convenient to place the terms of the dividend and the divisor, so that the exponents of the same letter shall either increase regularly, or diminish regularly, from the left to the right. This is termed, *arranging the dividend and divisor, with reference to a certain letter.* The letter with reference to which a quantity is arranged, is called *the letter of arrangement.*

The divisor is placed on the right of the dividend, because it is more easily multiplied by the respective terms of the quotient, as they are found.

From the preceding, we derive the

RULE,

FOR THE DIVISION OF ONE POLYNOMIAL BY ANOTHER.

Arrange the dividend and divisor, with reference to a certain letter and place the divisor on the right of the dividend.

Divide the first term of the dividend by the first term of the divisor, the result will be the first term of the quotient. Multiply the divisor by this term, and subtract the product from the dividend.

Divide the first term of the remainder by the first term of the divisor, the result will be the second term of the quotient. Multiply the divisor by this term, and subtract the product from the last remainder.

Proceed in the same manner, and if you obtain 0 *for a remainder, the division is said to be exact.*

Remarks.—1st. It is not *absolutely necessary* to arrange the dividend and divisor with reference to a certain letter; it should always be done, however, as a matter of convenience.

2d. The divisor may be placed on the *left* of the dividend, instead of the *right*, as directed in the rule. When the divisor is a monomial, it is more convenient to place it on the left; but, when it is a polynomial, to place it on the right.

3d. If there are more than two terms in the quotient, it is not necessary to bring down any more terms of the remainder, at each successive subtraction, than have corresponding terms in the quantity to be subtracted.

4th. It is a useful exercise for the learner, to perform the same example in two different ways. First, by arranging the dividend and divisor, so that the powers of the same letter shall *diminish* from left to right; and, secondly, so that the powers of the same letter shall *increase* from left to right.

5th. It is evident, that the exact division of one polynomial by another will be impossible, when the first term of the arranged dividend is not exactly divisible by the first term of the arranged divisor; or, when the first term of any of the remainders is not divisible by the first term of the divisor.

1. Divide $6a^2-13ax+6x^2$ by $2a-3x$.

$$\begin{array}{r|l}
6a^2-13ax+6x^2 & 2a-3x \\ \cline{2-2}
6a^2-9ax & 3a-2x \text{ Quotient.} \\ \cline{1-1}
-4ax+6x^2 & \\
-4ax+6x^2 & \\ \cline{1-1}
\end{array}$$

2. Divide x^2-y^2 by $x-y$.

$$\begin{array}{r|l}
x^2-y^2 & x-y \\ \cline{2-2}
x^2-xy & x+y \text{ Quotient.} \\ \cline{1-1}
xy-y^2 & \\
xy-y^2 & \\ \cline{1-1}
\end{array}$$

3. Divide a^3+x^3 by $a+x$.

$$\begin{array}{r|l}
a^3+x^3 & a+x \\ \cline{2-2}
a^3+a^2x & a^2-ax+x^2 \text{ Quot.} \\ \cline{1-1}
-a^2x+x^3 & \\
-a^2x-ax^2 & \\ \cline{1-1}
+ax^2+x^3 & \\
ax^2+x^3 & \\
\end{array}$$

4. Divide $5a^2x+5ax^2+a^3+x^3$ by $4ax+a^2+x^2$.

$$\begin{array}{l|l}
a^3+5a^2x+5ax^2+x^3 & a^2+4ax+x^2 \\
\underline{a^3+4a^2x+ax^2} & a+x \text{ Quotient.} \\
\quad a^2x+4ax^2+x^3 & \\
\quad \underline{a^2x+4ax^2+x^3} &
\end{array}$$

In this example, neither divisor nor dividend being arranged with reference to either a or x, we arrange them with reference to a, and then proceed to perform the division.

5. Divide $a^2+a^3-5a^4+3a^5$ by $a-a^2$.

Division performed, by arranging both quantities according to the ascending powers of a.

$$\begin{array}{l|l}
a^2+a^3-5a^4+3a^5 & a-a^2 \\
\underline{a^2-a^3} & a+2a^2-3a^3 \\
\quad +2a^3-5a^4 & \quad \text{Quotient.} \\
\quad \underline{2a^3-2a^4} & \\
\qquad -3a^4+3a^5 & \\
\qquad \underline{-3a^4+3a^5} &
\end{array}$$

Division performed, by arranging both quantities according to the descending powers of a.

$$\begin{array}{l|l}
3a^5-5a^4+a^3+a^2 & -a^2+a \\
\underline{3a^5-3a^4} & -3a^3+2a^2+a \\
\quad -2a^4+a^3 & \quad \text{Quotient.} \\
\quad \underline{-2a^4+2a^3} & \\
\qquad -a^3+a^2 & \\
\qquad \underline{-a^3+a^2} &
\end{array}$$

The pupil will perceive that the two quotients are the same, but differently arranged.

EXAMPLES.

6. Divide $4a^2-8ax+4x^2$ by $2a-2x$. . . . . . Ans. $2a-2x$.
7. Divide $2x^2+7xy+6y^2$ by $x+2y$. . . . . . . Ans. $2x+3y$.
8. Divide $2mx+3nx+10mn+15n^2$ by $x+5n$. . Ans. $2m+3n$.
9. Divide $x^2+2xy+y^2$ by $x+y$. . . . . Ans. $x+y$.
10. Divide $8a^4-8x^4$ by $2a^2-2x^2$. . . . . . . Ans. $4a^2+4x^2$.
11. Divide $ac+bc-ad-bd$ by $a+b$. . . . . . . Ans. $c-d$.
12. Divide $x^3+y^3+5xy^2+5x^2y$ by $x^2+4xy+y^2$. . . Ans. $x+y$.
13. Divide $a^3-9a^2+27a-27$ by $a-3$. . . . Ans. a^2-6a+9.
14. Divide $4a^4-5a^2x^2+x^4$ by $2a^2-3ax+x^2$. Ans. $2a^2+3ax+x^2$
15. Divide x^4-y^4 by $x-y$. . . . . . . Ans. $x^3+x^2y+xy^2+y^3$.

REVIEW.—79. In multiplying one polynomial by another, what terms in the product cannot be added together? How is the term of the quotient found, which contains the highest power of any particular letter? After obtaining the first remainder, how is the second term of the quotient found? What is understood by arranging the dividend and divisor with reference to a certain letter? What is the letter of arrangement? Why is the divisor placed on the right of the quotient? What is the rule for the division of one polynomial by another? When is the exact division of one polynomial by another impossible?

16. Divide a^3-b^3 by a^2+ab+b^2. . . . . Ans. $a-b$.
17. Divide $x^3-y^3+3xy^2-3x^2y$ by $x-y$. . . Ans. $x^2-2xy+y^2$.
18. Divide $4x^4-64$ by $2x-4$. . . . Ans. $2x^3+4x^2+8x+16$.
19. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.
Ans. $a^3-3a^2x+3ax^2-x^3$
20. Divide $4a^6-25a^2x^4+20ax^5-4x^6$ by $2a^3-5ax^2+2x^3$.
Ans. $2a^3+5ax^2-2x^3$
21. Divide y^3+1 by $y+1$. . . . . Ans. y^2-y+1
22. Divide $6a^4+4a^3x-9a^2x^2-3ax^3+2x^4$ by $2a^2+2ax-x^2$.
Ans. $3a^2-ax-2x^2$.
23. Divide $3a^4-8a^2b^2+3a^2c^2+5b^4-3b^2c^2$ by a^2-b^2.
Ans. $3a^2-5b^2+3c^2$.
24. Divide $x^6-3x^4y^2+3x^2y^4-y^6$ by $x^3-3x^2y+3xy^2-y^3$.
Ans. $x^3+3x^2y+3xy^2+y^3$.

MISCELLANEOUS EXERCISES.

1. $3a+5x-9c+7d+5a-3x-3d-(4a+2x-8c+4d)=$what?
Ans. $4a-c$.
2. $6ab-3cx+5d-ab+5cx-8d-(3ab+cx-3d)=$what?
Ans. $2ab+cx$.
3. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)=$what?
Ans. $7a-5b$.
4. $(a+b)(a+b)+(a-b)(a-b)=$what? . . . . Ans. $2a^2+2b^2$.
5. $(x+z)(x+z)-(x-z)(x-z)=$what? . . . . . . Ans. $4xz$.
6. $(a^2+a^4+a^6)(a^2-1)-(a^4+a)(a^4-a)=$what? . . . Ans. 0.
7. $(a^4+a^2z^2+z^4)\div(a^2-az+z^2)-(a+z)(a-z)=$what? A. $az+2z^2$
8. $(-1+a^3n^3)\div(-1+an)+(1+an)(1-an)=$what? A. $2+an$.
9. $(a^3+a^2b-ab^2-b^3)\div(a-b)-(a-b)(a-b)=$what? Ans. $4ab$.

CHAPTER II.

ALGEBRAIC THEOREMS.

DERIVED FROM MULTIPLICATION AND DIVISION.

Art. **80.**—If we square $a+b$, that is, multiply $a+b$ by itself, the product will be $a^2+2ab+b^2$; thus:

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

But $a+b$ is the sum of the quantities, a and b; hence

THEOREM I.

The square of the sum of two quantities, is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

EXAMPLES.

NOTE.—The instructor should read each of the following examples aloud, and require the pupil, by applying the theorem, to write at once the result on a slate, or blackboard. The examples may be enunciated thus: What is the square of $2a+b$?

1. $(2+3)^2=4+12+9=25$.
2. $(2a+b)^2=4a^2+4ab+b^2$.
3. $(2x+3y)^2=4x^2+12xy+9y^2$.
4. $(ab+cd)^2=a^2b^2+2abcd+c^2d^2$.
5. $(x^2+xy)^2=x^4+2x^3y+x^2y^2$.
6. $(2a^2+3ax)^2=4a^4+12a^3x+9a^2x^2$.

ART. 81.—If we square $a-b$, that is, multiply $a-b$ by itself, the product will be $a^2-2ab+b^2$. Thus:

$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\quad -ab+b^2} \\ a^2-2ab+b^2 \end{array}$$

But $a-b$ is the difference of the quantities a and b; hence

THEOREM II.

The square of the difference of two quantities, is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.

EXAMPLES.

1. $(5-4)^2=25-40+16=1$.
2. $(2a-b)^2=4a^2-4ab+b^2$.
3. $(3x-2y)^2=9x^2-12xy+4y^2$.
4. $(x^2-y^2)^2=x^4-2x^2y^2+y^4$.
5. $(ax-x^2)^2=a^2x^2-2ax^3+x^4$.
6. $(5a^2-b^2)^2=25a^4-10a^2b^2+b^4$.

ART. 82.—If we multiply $a+b$ by $a-b$, the product will be a^2-b^2. Thus:

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2-b^2 \end{array}$$

But $a+b$ represents the sum of two quantities, and $a-b$, their difference; hence,

THEOREM III.

The product of the sum and difference of two quantities, is equal to the difference of their squares.

EXAMPLES.

1. $(5+3)(5-3)=25-9=16=8\times2$.
2. $(2a+b)(2a-b)=4a^2-b^2$.
3. $(2x+3y)(2x-3y)=4x^2-9y^2$.
4. $(5a+4b)(5a-4b)=25a^2-16b^2$.
5. $(a^2+b^2)(a^2-b^2)=a^4-b^4$.
6. $(2am+3bn)(2am-3bn)=4a^2m^2-9b^2n^2$.

ART. 83.—If we divide a^3 by a^5, since the rule for the exponents requires that the exponent of the divisor should be subtracted from that of the dividend, we have $\frac{a^3}{a^5}=a^{3-5}=a^{-2}$.

But, since the value of a fraction is not altered by dividing both terms by the same quantity, (Art. 127), if we divide both numerator and denominator by a^3, we have $\frac{a^3}{a^5}=\frac{1}{a^2}$.

Hence $a^{-2}=\frac{1}{a^2}$, since each equals $\frac{a^3}{a^5}$.

In the same manner by subtracting the exponents

$$\frac{a^m}{a^n}=a^{m-n};$$

Or, by dividing both terms by a^m, $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$;

Hence, ... $a^{m-n}=\frac{1}{a^{n-m}}$. Therefore,

THEOREM IV.

The reciprocal of a quantity is equal to the same quantity with the sign of its exponent changed.

Thus, since $\frac{1}{a^m}$ is the reciprocal of a^m (Art. 51);

$$a^m=\frac{1}{a^{-m}}, \qquad \text{and } a^{-m}=\frac{1}{a^m}$$

Also . . . . $\frac{a}{b^m}=ab^{-m}$; $\qquad \frac{a^m}{b^n}=a^mb^{-n}$;

$$\frac{a}{b}=ab^{-1}; \qquad \frac{1}{ab^2}=a^{-1}b^{-2}.$$

From this we see, that *any factor may be transferred from one term of a fraction to the other, if, at the same time, the sign of its exponent be changed.*

Thus: . . . . $\frac{a}{b}=ab^{-1}=\frac{b^{-1}}{a^{-1}}=\frac{1}{a^{-1}b}$;

$$\frac{a^2b^2}{c^2d^3}=a^2b^2c^{-2}d^{-3}=\frac{1}{a^{-2}b^{-2}c^2d^3}=\frac{c^{-2}d^{-3}}{a^{-2}b^{-2}}.$$

ART. **84.**—Let it be required to divide a^2 by a^2. By the rule for the exponents, (Art. 74), $\frac{a^2}{a^2}=a^{2-2}=a^0$; but since any quantity is contained in itself once, $\frac{a^2}{a^2}=1$.

Similarly, $\frac{a^m}{a^m}=a^{m-m}=a^0$; but $\frac{a^m}{a^m}=1$, therefore $a^0=1$ since each is equal to $\frac{a^m}{a^m}$. Hence,

THEOREM V.

Any quantity whose exponent is 0 is equal to unity.

This notation is used, when we wish to preserve the trace of a letter, which has disappeared in the operation of division. Thus, if it is required to divide m^2n^2 by mn^2, the quotient will be $\frac{m^2n^2}{mn^2}=m^{2-1}n^{2-2}=m^1n^0=m$, since $n^0=1$. Now, the quotient is correctly expressed either by m^1n^0, or m, since both have the same value. The first form is used, when it is necessary to show that n originally entered as a factor into the dividend and divisor.

ART. **85.**—1. If we divide a^2-b^2 by $a-b$, the quotient will be $a+b$.

2. If we divide a^3-b^3 by $a-b$, the quotient will be a^2+ab+b^2.

In the same manner, we would find, by trial, that the difference of the same powers of two quantities, is always divisible by the difference of the quantities. The direct proof of this theorem is as follows.

Let us divide a^m-b^m by $a-b$.

$$\begin{array}{l|l}
a^m-b^m & a-b \\
\hline
a^m-a^{m-1}b & a^{m-1}+\dfrac{b(a^{m-1}-b^{m-1})}{a-b} \text{ Quotient.} \\
\hline
\quad a^{m-1}b-b^m & \\
\quad =b(a^{m-1}-b^{m-1}) \text{ Remainder.} &
\end{array}$$

In performing this division, we see that the first term of the quotient is a^{m-1}, and that the first remainder is $b(a^{m-1}-b^{m-1})$.

The remainder consists of two factors, b and $a^{m-1}-b^{m-1}$. Now, it is evident, that *if* the second of these factors is divisible by $a-b$, then will the quantity a^m-b^m be divisible by $a-b$. Thus, if $a-b$ is contained c times in $a^{m-1}-b^{m-1}$, the whole quotient of a^m-b^m, divided by $a-b$, would be $a^{m-1}+bc$

From this, we see that IF $a^{m-1}-b^{m-1}$ is divisible by $a-b$, then will a^m-b^m be also divisible by it. That is, IF *the difference of the same powers of two quantities is divisible by the difference of the quantities themselves, then will the difference of the next higher powers of the same quantities, be divisible by the difference of the quantities.* But we have seen, already, that a^2-b^2 is divisible by $a-b$; hence, it follows, that a^3-b^3 is also divisible by $a-b$. Then, since a^3-b^3 is divisible by $a-b$, it again follows, that a^4-b^4 is divisible by it; and so on, without limit. Hence, we have

THEOREM VI.

The difference of the same powers of two quantities, is always divisible by the difference of the quantities.

The quotients obtained by dividing the difference of the same powers of two quantities, by the difference of those quantities, follow a simple law. Thus:

$$(a^2-b^2)\div(a-b)=a+b.$$
$$(a^3-b^3)\div(a-b)=a^2+ab+b^2.$$
$$(a^4-b^4)\div(a-b)=a^3+a^2b+ab^2+b^3.$$
$$(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4.$$

The exponent of the first letter decreases by unity, while that of the second increases by unity.

ART. **86.**—Since a^m-b^m is always divisible by $a-b$, if we put $-c$ for b, then $a-b$ will become $a+c$, and, since b^m will become c^m, when m is even, as 2, 4, 6, &c., and $-c^m$, when m is odd, as 3, 5, 7, &c., therefore, a^m-b^m will become a^m-c^m, when m is even, and a^m+c^m, when m is odd, because $a^m-b^m=a^m-(-c^m)=a^m+c^m$; therefore, a^m-c^m is always divisible by $a+c$, when m is even, and a^m+c^m is always divisible by $a+c$ when m is odd. These truths are expressed in the following theorems.

THEOREM VII.

The difference of the even powers of the same degree of two quantities, is always divisible by the sum of the quantities.

Thus: $(a^2-b^2)\div(a+b)=a-b.$

$$(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3.$$
$$(a^6-b^6)\div(a+b)=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5.$$

REVIEW.—80. To what is the square of the sum of two quantities equal? 81. To what is the square of the difference of two quantities equal? 82. To what is the product of the sum and difference of two quantities equal? 83. How may the reciprocal of any quantity be expressed? How may any factor be transferred from one term of a fraction to the other? In what other form may a^m be written? a^{-m}? 84. What is the value of any quantity whose exponent is zero?

THEOREM VIII.

The sum of the odd powers of the same degree of two quantities, is always divisible by the sum of the quantities.

Thus: $(a^3+b^3)\div(a+b)=a^2-ab+b^2$.

$(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$.

$(a^7+b^7)\div(a+b)=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.

FACTORING.

FACTORS, AND DIVISORS OF ALGEBRAIC QUANTITIES.

ART. **87.**—A *divisor* or *measure* of a quantity, is any quantity that divides it without a remainder, or that is exactly contained in it. Thus, 2 is a divisor of 6; and a^2 is a divisor or measure of a^2x.

ART. **88.**—A *prime number*, is one which has no divisors except itself and unity.

A *composite number*, is one which has one or more divisors besides itself and unity.

Hence all numbers are either prime or composite; and every composite number is the product of two or more prime numbers.

The following is a list of the prime numbers under 100:

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

The composite numbers are, 4, 6, 8, 9, 10, 12, &c.

RULE,

FOR RESOLVING ANY COMPOSITE NUMBER INTO ITS PRIME FACTORS.

Divide by any prime number that will exactly divide it; divide the quotient again in the same manner; and so continue to divide, until a quotient is obtained, which is a prime number; then, the last quotient and the several divisors, will constitute the prime factors of the given number.

REMARK.—The reason of this rule is evident, from the nature of prime and composite numbers. It will be found most convenient to divide first by the smallest prime number that is a factor.—See Ray's Arithmetic, Part III., FACTORING.

REVIEW.—85. By what is the difference of the same powers of two quantities always divisible? 86. By what is the difference of the even powers of the same degree of two quantities always divisible? By what is the sum of the odd powers of the same degree of two quantities always divisible?

EXAMPLES.

1. The composite numbers under 100, that is, 4, 6, 8, &c., may be given as examples. Every pupil should learn to give the factors of these quantities readily.

2. What are the prime factors of 105? . . . . . Ans. 3, 5, 7.
3. What are the prime factors of 210? . . . Ans. 2, 3, 5, 7.
4. Resolve 4290 into its prime factors. . Ans. 2, 3, 5, 11, 13.

Art. **89.**—A *prime quantity*, in Algebra, is one which is exactly divisible only by itself and by unity. Thus, a, b, and $b+c$ are prime quantities; while ab and $ab+ac$ are not prime.

Art. **90.**—Two quantities, like two numbers, are said to be *prime to each other*, or *relatively prime*, when no quantity except unity will exactly divide them both. Thus, ab and cd are prime to each other.

Art. **91.**—A *composite number*, or a *composite quantity*, is one which is the product of two or more factors, neither of which is unity. Thus, ax is a composite quantity, of which the factors are a and x.

Remark.—A monomial may be a composite quantity, as ax; and a polynomial may not be a composite quantity, as a^2+x^2.

Art. **92.**—To separate a monomial into its prime factors.

RULE.

Resolve the coëfficient into its prime factors; then these, with the literal factors of the monomials, will form the prime factors of the given quantity. The reason of this rule is self-evident.

Find the prime factors of the following nominals:

1. $15a^2bc$. . . . . Ans. $3\times5.a.a.b.c$.
2. $21ab^2d$. . . . . Ans. $3\times7.a.b.b.d$.
3. $35abc^2x$. . . . . Ans. $5\times7.a.b.c.c.x$.
4. $39a^2m^2n$. . . . . Ans. $3\times13.a.a.m.m.n$.

Art. **93.**—To separate a polynomial into its factors, when one of them is a monomial and the other a polynomial.

Review.—87. What is the divisor of a quantity? 88. What is a prime number? What is a composite number? Name several of the prime numbers, beginning with unity. Name several of the composite numbers, beginning with 4. What is the rule for resolving any composite number into its prime factors? 89. What is a prime quantity? Give an example. 90. When are two quantities prime to each other? Give an example. 91. What is a composite quantity? Give an example. 92. What is the rule for separating a monomial into its prime factors?

RULE.

Divide the given quantity by the greatest monomial that will exactly divide each of its terms. Then the monomial divisor will be one factor, and the quotient the other. The reason of this rule is self-evident.

Separate the following expressions into factors:

1. $x+ax$. . . . . Ans. $x(1+a)$.
2. $am+ac$. . . . . Ans. $a(m+c)$.
3. bc^2+bcd. . . . . Ans. $bc(c+d)$.
4. $4x^2+6xy$. . . . . Ans. $2x(2x+3y)$.
5. $6ax^2y+9bxy^2-12cx^2y$. . . . . Ans. $3xy(2ax+3by-4cx)$.
6. $5ax^2-35ax^3y+5a^2x^3y$. . . . . . Ans. $5ax^2(1-7xy+axy)$.
7. $14a^3x^2y+21a^2x^2y^2-35a^3xy^2$. . Ans. $7a^2xy(2ax+3xy-5ay)$.
8. $6bc^2x-15bc^3-3b^2c^2$. . . . . Ans. $3bc^2(2x-5c-b)$.
9. $a^3cm^2+a^2c^2m^2-a^2cm^3$. . . . . Ans. $a^2cm^2(a+c-m)$.

Art. **94.**—To separate a quantity which is the product of two or more polynomials, into its prime factors.

No general rule can be given, for this case. When the given quantity does not consist of more than three terms, the pupil will generally be able to accomplish it, if he is familiar with the theorems in the preceding section.

1st. Any trinomial can be separated into two binomial factors, when the extremes are squares and positive, and the middle term is twice the product of the square roots of the extremes. See Articles 80 and 81.

Thus: $a^2+2ab+b^2=(a+b)(a+b)$.
$a^2-2ab+b^2=(a-b)(a-b)$.

2d. Any binomial, which is the difference of two squares, can be separated into two factors, one of which is the sum, and the other the difference of the roots. See Art. 82.

Thus: $a^2-b^2=(a+b)(a-b)$.

3d. When any expression consists of the difference of the same powers of two quantities, it can be separated into at least two factors, one of which is the difference of the quantities. See Art. 85.

Thus: $a^m-b^m=(a-b)(a^{m-1}+a^{m-2}b \; . \; . \; . \; . \; . \; +ab^{m-2}+b^{m-1})$, where a, b, and m, may be any quantities whatever.

In this case, one of the factors being the difference of the quantities, the other will be found by dividing the given expression by this difference. Thus, to find the other factor of a^3-b^3, divide by $a-b$, the quotient will be found to be a^2+ab+b^2; hence, $a^3-b^3=(a-b)(a^2+ab+b^2)$.

In a similar manner, $a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.

4th. When any expression consists of the difference of the even powers of two quantities, higher than the second degree, it can be separated into at least three factors, one of which is the sum, and another the difference of the quantities. See Articles 85 and 86.

Thus, a^4-b^4 is exactly divisible by $a+b$, according to Article 86; and, according to Article 85, it is exactly divisible by $a-b$; hence, it is exactly divisible by both $a+b$ and $a-b$; and the other factor will be found by dividing by their product. Or, it may be separated into factors, according to paragraph 2d, above, thus:

$$a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b).$$

5th. When any expression consists of the sum of the odd powers of two quantities, it may be separated into at least two factors, one of which is the sum of the quantities (See Art. 86). The other factor will be found, by dividing the given expression by this sum. Thus, we know that a^3+b^3, is exactly divisible by $a+b$, and by division, we find the other factor to be a^2-ab+b^2; hence, $a^3+b^3=(a+b)(a^2-ab+b^2)$.

Separate the following expressions into their simplest factors

1. $x^2+2xy+y^2$.
2. $9a^2+12ab+4b^2$.
3. $4+12x+9x^2$.
4. $m^2-2mn+n^2$.
5. $a^2-2abx+b^2x^2$.
6. $4x^2-20xz+25z^2$.
7. x^2-y^2.
8. $9m^2-16n^2$.
9. $a^2b^2-c^2d^2$.
10. a^2x-x^3.
11. x^4-b^4.
12. y^3+1.
13. x^3-1.
14. $8a^3-27b^3$.
15. a^5+b^5.
16. a^6-b^6.

ANSWERS.

1. $(x+y)(x+y)$.
2. $(3a+2b)(3a+2b)$.
3. $(2+3x)(2+3x)$.
4. $(m-n)(m-n)$.
5. $(a-bx)(a-bx)$.
6. $(2x-5z)(2x-5z)$.
7. $(x+y)(x-y)$.
8. $(3m+4n)(3m-4n)$.
9. $(ab+cd)(ab-cd)$.
10. $x(a+x)(a-x)$.
11. $(x^2+b^2)(x^2-b^2)=(x^2+b^2)(x+b)(x-b)$.
12. $(y+1)(y^2-y+1)$.
13. $(x-1)(x^2+x+1)$.
14. $(2a-3b)(4a^2+6ab+9b^2)$.
15. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$.
16. $(a^3+b^3)(a^3-b^3)=(a^3+b^3)(a-b)(a^2+ab+b^2)$.
$=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.
$=(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)$.
$=(a^2-b^2)(a^4+a^2b^2+b^4)$.

ART. **95.**—To separate a quadratic trinomial into its factors.

A quadratic trinomial is of the form, x^2+ax+b, in which the signs of the second and third terms may be either plus or minus. When this operation is practicable, the method of doing it, may be learned by observing the relation that exists between two binomial factors and their product.

1. $(x+a)(x+b)=x^2+(a+b)x+ab$.
2. $(x-a)(x-b)=x^2-(a+b)x+ab$.
3. $(x+a)(x-b)=x^2+(a-b)x-ab$.
4. $(x-a)(x+b)=x^2+(b-a)x-ab$.

From the preceding, we see, that when the first term of a quadratic trinomial is a square, with the coëfficient of its second term equal to the sum of any two quantities, which, being multiplied together, will produce the third term, it may be resolved into two binomial factors by inspection.

Decompose each of the following trinomials into two binomial factors.

1. x^2+5x+6. . . . . Ans. $(x+2)(x+3)$.
2. $a^2+7a+12$. . . . . Ans. $(a+3)(a+4)$.
3. x^2-5x+6. . . . . Ans. $(x-2)(x-3)$.
4. $x^2-9x+20$. . . . . Ans. $(x-4)(x-5)$.
5. x^2+x-6. . . . . Ans. $(x+3)(x-2)$.
6. x^2-x-6. . . . . Ans. $(x-3)(x+2)$.
7. x^2+x-2. . . . . Ans. $(x+2)(x-1)$.
8. $x^2-13x+40$. . . . . Ans. $(x-8)(x-5)$.
9. x^2-7x-8. . . . . Ans. $(x-8)(x+1)$.
10. $x^2+7x-18$. . . . . Ans. $(x+9)(x-2)$.
11. x^2-x-30. . . . . Ans. $(x-6)(x+5)$.

In the same manner, we may often separate other trinomials into factors, by first taking out the monomial factor common to each term.

Thus, $5ax^2-10ax-40a=5a(x^2-2x-8)=5a(x-4)(x+2)$.

12. $3x^2+12x-15$. . . . . Ans. $3(x+5)(x-1)$.
13. $a^2x^2-9a^2x+14a^2$. . . . . Ans. $a^2(x-7)(x-2)$.
14. $2abx^2-14abx-60ab$. . . . . . Ans. $2ab(x-10)(x+3)$.
15. $2x^3-4x^2-30x$. . . . . Ans. $2x(x-5)(x+3)$.

REVIEW.—93. What is the rule for separating a polynomial into its prime factors, when one of them is a monomial, and the other a polynomial? 94. When can a trinomial be separated into two binomial factors? What are the factors of $m^2+2mn+n^2$? Of $c^2-2cd+d^2$? When can a binomial be separated into two binomial factors? What are the factors of x^2-y^2? Of $9a^2-16b^2$? What is one of the factors of a^2-b^2? Of a^3-b^3? Of x^4-y^4? What are two of the factors of a^4-b^4? Of a^6-b^6?

ART. **96.**—The principal use of factoring, is to shorten the work, and simplify the results of algebraic operations. Thus, when it is required to multiply and divide by algebraic expressions, if the multiplier and divisor contain a common factor, it may be canceled, or left out in both, without affecting the value of the result. Thus, if it is required to multiply any quantity by a^2-b^2, and then to divide the product by $a+b$, the result will be the same as to multiply at once by $a-b$.

Whenever there is an opportunity of canceling common factors, the operations to be performed should be merely indicated, as the common factors will then be more easily discovered. The pupil will see the application of this principle, by solving the following examples.

1. Multiply $a-b$ by $x^2+2xy+y^2$, and divide the product by $x+y$.

$$\frac{(a-b)(x^2+2xy+y^2)}{x+y}=\frac{(a-b)(x+y)(x+y)}{x+y}=(a-b)(x+y)$$
$$=ax+ay-bx-by.$$

2. Multiply $x-3$ by x^2-1, and divide the product by $x-1$, by factoring. Ans. x^2-2x-3.

3. Divide z^3+1 by $z+1$, and multiply the quotient by z^2-1, by factoring. Ans. z^4-z^3+z-1.

4. Divide $6a^2c-12abc+6b^2c$ by $2ac-2bc$, by factoring. Ans. $3(a-b)$.

5. Multiply $6ax+9ay$ by $4x^2-9y^2$, and divide the product by $4x^2+12xy+9y^2$, by factoring. Ans. $3a(2x-3y)$.

6. Multiply x^2-5x+6 by $x^2-7x+12$, and divide the quotient by x^2-6x+9, by factoring. Ans. $(x-2)(x-4)$.

Other examples in which the principle may be applied, will be found in the multiplication and division of fractions.

GREATEST COMMON DIVISOR.

ART. **97.**—ANY quantity that will exactly divide two or more quantities, is called a *common divisor*, or *common measure*, of those quantities. Thus, 2 is a common divisor of 8 and 12; and a is a common divisor of ab and a^2x.

REMARK.—Two quantities may sometimes have more than one common divisor. Thus, 8 and 12 have two common divisors, 2 and 4.

REVIEW.—94. What is one of the factors of a^3+b^3? What is one of the factors of x^5+y^5? 95. What is a quadratic trinomial?

ART. **98.**—That common divisor of two quantities, which is the greatest, both with regard to the coëfficients and exponents, is called their *greatest common divisor*, or *greatest common measure.* Thus, the greatest common divisor of $4a^2xy$ and $6a^3x^2y^2$ is $2a^2xy$.

ART. **99.**—Quantities that have a common divisor, are said to be *commensurable;* and those that have no common divisor, are said to be *incommensurable.* Incommensurable quantities are also said to be *prime* to each other, or *relatively prime.*

ART. **100.**—To find the greatest common divisor of two or more monomials.

1. Let it be required to find the greatest common divisor of the two monomials, $6ab$ and $15a^2c$.

By separating each quantity into its prime factors, we have $6ab=2\times3ab$, $15a^2c=3\times5aac$.

Here we see, that 3 and a are the only factors common to both terms; hence, both the quantities can be exactly divided, either by 3 or a, or by their product $3a$, and by no other quantity whatever; consequently, $3a$ is their greatest common divisor Hence, the

RULE,

FOR FINDING THE GREATEST COMMON DIVISOR OF TWO OR MORE MONOMIALS.

Resolve the quantities into their prime factors; then, the product of those factors that are common to each of the terms, will form the greatest common divisor.

NOTE.—The greatest common divisor of the literal parts of the quantities, may generally be more easily found by inspection, by taking each letter with the highest power, that is common to all the quantities.

2. Find the greatest common divisor of $4a^2x^3$, $6a^3x^2$, and $10a^4x$.

$4a^2x^3=2\times2a^2x^3$
$6a^3x^2=2\times3a^3x^2$
$10a^4x=2\times5a^4x$

Here we see, that 2, a^2, and x are the only factors common to all the quantities; hence, $2a^2x$ is the greatest common divisor.

Find the greatest common divisor of the following quantities.

3. $4a^2x^2$, and $10ax^3$. . . . . Ans. $2ax^2$.
4. $9abc^3$, and $12bc^4x$. . . . . Ans. $3bc^3$.
5. $4a^3b^2x^5y^3$, and $8a^5x^2y^2$. . . . . Ans. $4a^3x^2y^2$.
6. $3a^4y^3$, $6a^5x^3y^5$, and $9a^6y^4z$. . . . . Ans. $3a^4y^3$.
7. $8ax^2y^4z^5$, $12x^5z^6$, and $24a^3x^3z^2$. . . . . Ans. $4x^2z^2$.
8. $6a^2xy^2$, $12a^3y^4z^5$, $9a^5x^3y^4$, and $24a^3y^6z$. . . . . Ans. $3a^2y^2$.

ART. **101.**—To find the greatest common divisor of two polynomials.

First. Let AD and BD be either two monomials, or polynomials, of which D is a common divisor; and let AD be greater than BD.

Divide AD by BD; then, if it gives an exact quotient, BD must be the greatest common divisor, since no quantity can have a divisor greater than itself. But, if BD is not contained an exact number of times in AD, suppose it is contained Q times with a remainder, which may be called R. Then, since the remainder is found, by subtracting the product of the divisor by the quotient, from the dividend, we have R=AD—BDQ.

$$\begin{array}{l} BD)AD(Q \\ \underline{BDQ} \\ AD-BDQ=R \end{array}$$

Dividing both sides by D, we get $\frac{R}{D}=A-BQ$; But A and BQ are each entire quantities; hence $\frac{R}{D}$, which is equal to their difference, must be an entire quantity. Hence, it follows, that *any common divisor of two quantities, will always exactly divide their remainder after division.* And, since the greatest common divisor is *a* common divisor, it follows that the *greatest common divisor* of two quantities, will always exactly divide their remainder after division.

REMARK.—In the above article, we have used two axioms, which may be new to some pupils. They are, first: *If two equal quantities be divided by the same quantity, their quotients will be equal.* And, second: *The difference of two entire quantities is also an entire quantity.* The pupil can easily see, that the sum, or difference of two whole numbers must also be a whole number; and, that the same is likewise true of two entire quantities. This, and the next article will both be better understood by the pupil, after he has studied simple equations.

ART. **102.**—Second. Suppose, now, that it is required to find the greatest common divisor of two polynomials, A and B, of which A is the greater.

If we divide A by B, and there is no remainder, B is, evidently, the greatest common divisor, since it can have no divisor greater than itself.

$$\begin{array}{l} B)A(Q \\ \underline{BQ} \\ A-BQ=R\text{, 1st Rem.} \end{array}$$

Dividing A by B, and calling the quotient Q, if there is a remainder R, it is evidently less than either of the quantities A and B; and, by the preceding theorem, it is also exactly divisible by the greatest common divisor; hence, the greatest common divisor must divide A, B, and R, and can not be greater than R. But if R will exactly divide B, it will also exactly divide A, since A=BQ+R, and will be the greatest common divisor sought.

$$\begin{array}{l} R)B(Q' \\ \underline{RQ'} \\ B-RQ'=R'\text{, 2d Rem.} \end{array}$$

$A=BQ+R$
$B=RQ'+R'$

Since the dividend is equal to the product of the divisor by the quotient, plus the remainder.

Suppose, however, that when we divide R into B, to ascertain if it will exactly divide it, we find that the quotient is Q', with a remainder, R'. Now, it has been shown, that whatever exactly divides two quantities, will divide their remainder after division; then, since the greatest common divisor of A and B, has been shown to divide B and R, it will also divide their remainder R', and can not be greater than R'. And, if R' exactly divides R, it will also divide B, since $B=RQ'+R'$; and whatever exactly divides B and R, will also exactly divide A, since $A=BQ+R$; therefore, if R' exactly divides R, it will exactly divide both A and B, and will be their greatest common divisor.

In the same manner, by continuing to divide the last divisor by the last remainder, it may always be shown, that the greatest common divisor of A and B will exactly divide every new remainder, and, of course, can not be greater than either of them. It may, also, always be shown, as above, in the case of R', that any remainder, which exactly divides the preceding divisor, will also exactly divide A and B. Then, since the greatest common divisor of A and B can not be greater than this remainder, and, as this remainder is a common divisor of A and B, it will be their greatest common divisor sought.

To illustrate the same principle by numbers, let it be required to find the greatest common divisor of 14 and 20.

If we divide 20 by 14, and there is no remainder, 14 is, evidently, the greatest common divisor, since it can have no divisor greater than itself Dividing 20 by 14, we find the quotient is 1, and the remainder 6, which is, necessarily, less than either of the quantities, 20 and 14; and by the theorem, Article 101, it is exactly divisible by their greatest common divisor; hence, the greatest common divisor must divide 20, 14, and 6, and cannot be greater than 6. Now, if 6 will exactly divide 14, it will also exactly divide 20, since $20=14+6$, and will be the greatest common divisor sought.

$$\begin{array}{l}
14)20(1 \\
\quad\ \underline{14} \\
\quad\quad 6)14(2 \\
\quad\quad\quad\ \underline{12} \\
\quad\quad\quad\quad 2)6(3 \\
\quad\quad\quad\quad\quad\ \underline{6}
\end{array}$$

But when we divide 6 into 14, to ascertain if it will exactly divide it, we find that the quotient is 2, with a remainder, 2; then,

REVIEW.—95. When can a quadratic trinomial be separated into binomial factors? 96. What is the principal use of factoring? 97. What is a common divisor of two or more quantities? Give an example. 98. What is the greatest common divisor of two quantities? Give an example. 99. When are quantities commensurable? When are quantities incommensurable? 100. How do you find the greatest common divisor of two or more monomials? 101. Prove that any common divisor of two quantities will always exactly divide their remainder, after division.

by the preceding theorem, the greatest common divisor of 14 and 6 will also divide 2, and therefore, can not be greater than 2. Now, if 2 will exactly divide 6, it will, also, exactly divide 14, since $14=6\times2+2$; and whatever will exactly divide 6 and 14, will also divide 20. But 2 exactly divides 6; hence it is the greatest common divisor of 14 and 20.

ART. **103.**—When the remainders decrease to unity, or when we arrive at a remainder which does not contain the letter of arrangement, we conclude that there is no common divisor to the quantities.

ART. **104.**—If one of the quantities contains a factor not found in the other, it may be canceled without affecting the common divisor (see example 3); and if both quantities contain a common factor, it may be set aside as a factor of the common divisor; and we may proceed to find the greatest common divisor of the other factors of the given quantities. This is self-evident. See Example 2.

ART. **105.**—We may multiply either quantity, by a factor not found in the other, without affecting the greatest common divisor.

Thus, in the fraction $\frac{2abx}{3abc}$, the greatest common divisor of the two terms, is evidently ab. Here, we may cancel the factors 2 and x in the numerator, or 3 and c in the denominator, without affecting the common divisor; for the common divisor of $\frac{ab}{3abc}$, or of $\frac{2abx}{ab}$, is still ab.

If we multiply the dividend by 4, a factor not found in the divisor, we have $\frac{8abx}{3abc}$, of which the common divisor is still ab.

In the same manner we may multiply the divisor by any factor not found in the dividend, and the common divisor will still remain the same.

If, however, we multiply the numerator by 3, which is a factor of the denominator, the result is $\frac{6abx}{3abc}$, of which the greatest common divisor is $3ab$, and not ab as before. Hence, we see, that the greatest common divisor will be changed, by multiplying one of the quantities by a factor of the other.

REVIEW.—102. Show, that by dividing the last divisor by the last remainder, the greatest common divisor of two polynomials will exactly divide both the first and second remainders after division.

ART. **106.**—In the general demonstration, Art. 101, it has been shown, that the greatest common divisor of two quantities, also exactly divides each of the successive remainders; hence, the preceding principles apply to the successive remainders that arise, in the course of the operations necessary to find the greatest common divisor.

The preceding principles will be illustrated by some examples.

1. Find the greatest common divisor of x^3-y^3 and $x^4-x^2y^2$.

Here the second quantity contains x^2 as a factor, but it is not a factor of the first; we may, therefore, cancel it, and the second quantity becomes x^2-y^2. Divide the first by it.

After dividing, we find that y^2 is a factor of the remainder, but not of x^2-y^2, the dividend. Hence, by canceling it, the divisor becomes $x-y$; then, dividing by this, we find there is no remainder; therefore $x-y$ is the greatest common divisor.

$$\begin{array}{l|l} x^3-y^3 & x^2-y^2 \\ \cline{2-2} \underline{x^3-xy^2} & (x \\ \quad xy^2-y^3 & \\ \text{or, } (x-y)y^2 & \end{array}$$

$$\begin{array}{l|l} x^2-y^2 & x-y \\ \cline{2-2} \underline{x^2-xy} & (x+y \\ \quad xy-y^2 & \\ \quad \underline{xy-y^2} & \end{array}$$

2. Find the greatest common divisor of $x^6+a^3x^3$ and $x^4-a^2x^2$.

The factor x^2 is common to both these quantities; it therefore forms part of the greatest common divisor, and may be taken out and reserved. Doing this, the quantities become x^4+a^3x and x^2-a^2. The first quantity still contains a common factor, x, which the latter does not; canceling this, it becomes x^3+a^3. Then, proceeding as in the first example, we find the greatest common divisor is $x^2(x+a)$.

$$\begin{array}{l|l} x^3+a^3 & x^2-a^2 \\ \cline{2-2} \underline{x^3-a^2x} & (x \\ \quad a^2x+a^3 & \\ \text{or, } (x+a)a^2 & \end{array}$$

$$\begin{array}{l|l} x^2-a^2 & x+a \\ \cline{2-2} \underline{x^2+ax} & (x-a \\ \quad -ax-a^2 & \\ \quad \underline{-ax-a^2} & \end{array}$$

3. Find the greatest common divisor of $5a^5+10a^4x+5a^3x^2$ and $a^3x+2a^2x^2+2ax^3+x^4$.

Here $5a^3$ is a factor of the first quantity only, and x, of the second only. Suppressing these factors, and proceeding as in the previous examples, we find $a+x$ is the greatest common divisor.

$$\begin{array}{l|l} a^3+2a^2x+2ax^2+x^3 & a^2+2ax+x^2 \\ \cline{2-2} \underline{a^3+2a^2x+ax^2} & (a \\ \quad ax^2+x^3 & \\ \text{or, } (a+x)x^2 & \end{array}$$

$$\begin{array}{l|l} a^2+2ax+x^2 & a+x \\ \cline{2-2} \underline{a^2+ax} & (a+x \\ \quad ax+x^2 & \\ \quad \underline{ax+x^2} & \end{array}$$

4. Find the greatest common divisor of $2a^4-a^2x^2-6x^4$ and $4a^5+6a^3x^2-2a^2x^3-3x^5$.

In solving this example, there are two instances in which it is necessary to multiply the dividend, in order that the coëfficient of the first term may be exactly divisible by the divisor. See Art. 105. The greatest common divisor is found to be $2a^2+3x^2$.

$$
\begin{array}{l|l}
4a^5+6a^3x^2-2a^2x^3-3x^5 & 2a^4-a^2x^2-6x^4 \\
\underline{4a^5-2a^3x^2-12ax^4} & (2a \\
\quad 8a^3x^2-2a^2x^3+12ax^4-3x^5 & \\
\text{or, } (8a^3-2a^2x+12ax^2-3x^3)x^2 &
\end{array}
$$

$$
\begin{array}{l}
2a^4-a^2x^2-6x^4 \\
\qquad\qquad 4 \\
\hline
\end{array}
$$

$$
\begin{array}{l|l}
8a^4-4a^2x^2-24x^4 & 8a^3-2a^2x+12ax^2-3x^3 \\
\underline{8a^4-2a^3x+12a^2x^2-3ax^3} & (a \\
\quad 2a^3x-16a^2x^2+3ax^3-24x^4 & \\
\qquad\qquad\qquad\qquad 4 & \\
\hline
\quad 8a^3x-64a^2x^2+12ax^3-96x^4 & (x \\
\quad \underline{8a^3x-\ \ 2a^2x^2+12ax^3-\ \ 3x^4} & \\
\qquad -62a^2x^2 \text{———} -93x^4 & \\
\quad \text{or, } -31x^2(2a^2+3x^2) &
\end{array}
$$

$$
\begin{array}{l|l}
8a^3-2a^2x+12ax^2-3x^3 & 2a^2+3x^2 \\
\underline{8a^3 \qquad\quad +12ax^2} & (4a-x \\
\quad -2a^2x \text{————} -3x^3 & \\
\quad \underline{-2a^2x \text{————} -3x^3} &
\end{array}
$$

From the preceding demonstrations and examples, we derive the

RULE,

FOR FINDING THE GREATEST COMMON DIVISOR OF TWO POLYNOMIALS.

1st. *Divide the greater polynomial by the less, and if there is no remainder, the less quantity will be the divisor sought.*

2d. *If there is a remainder, divide the first divisor by it, and continue to divide the last divisor by the last remainder, until a divisor is obtained, which leaves no remainder; this will be the greatest common divisor of the two given polynomials.*

REMARKS.—102. Explain the principles used, in finding the greatest common divisor, by finding it for the numbers 14 and 20. 103. When do we conclude that there is no common divisor to two quantities? 104. How is the common divisor of two quantities affected, by canceling a factor in one of them, not found in the other? When both quantities contain a common factor, how may it be treated? 105. How is the greatest common divisor of two quantities affected, by multiplying either of them by a factor not found in the other? 106. What is the rule for finding the greatest common divisor of two polynomials? How do you find the greatest common divisor of three or more quantities?

NOTES.—1. When the highest power of the *leading* letter is the same in both, it is immaterial which of the quantities is made the dividend.

2. If both quantities contain a common factor, let it be set aside, as forming a factor of the common divisor, and proceed to find the greatest common divisor of the remaining factors, as in Example 2.

3. If either quantity contains a factor not found in the other, it may be canceled, before commencing the operation, as in Example 3. See Art. 104.

4. Whenever it becomes necessary, the dividend may be multiplied by any quantity which will render the first term exactly divisible by the divisor. See Art. 105.

5. If, in any case, the remainder does not contain the leading letter, that is, if it is independent of that letter, there is no common divisor.

6. To find the greatest common divisor of three or more quantities, first find the greatest common divisor of two of them; then, of that divisor and one of the other quantities, and so on. The last divisor thus found, will be the greatest common divisor sought.

7. Since the greatest common divisor of two or more quantities contains all the factors common to these quantities, it may be found most easily by separating the quantities into factors, where this can be done, by means of the rules in the preceding article.

Find the greatest common divisor of the following quantities.

5. $5a^2+5ax$ and a^2-x^2. . . . . Ans. $a+x$.
6. x^3-a^2x and x^3-a^3. . . . . Ans. $x-a$.
7. x^3-c^2x and $x^2+2cx+c^2$. . . . . Ans. $x+c$.
8. x^2+2x-3 and x^2+5x+6. . . . . Ans. $x+3$.
9. $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$. . . . Ans. $2a+3x$.
10. a^4-x^4 and $a^3+a^2x-ax^2-x^3$. . . . . Ans. a^2-x^2.
11. $a^2-5ax+4x^2$ and $a^3-a^2x+3ax^2-3x^3$. . . . . . Ans. $a-x$.
12. $a^2x^4-a^2y^4$ and $x^5+x^3y^2$. . . . . Ans. x^2+y^2.
13. a^5-x^5 and $a^{13}-x^{13}$. . . . . Ans. $a-x$.

LEAST COMMON MULTIPLE.

ART. **107.**—A *multiple* of a quantity is that which contains it exactly. Thus, 6 is a multiple of 2, or of 3; and 24 is a multiple of 2, 3, 4, &c.; also, $8a^2b^3$ is a multiple of $2a$, of $2a^2$, of $2a^2b$, &c.; and $4(a-x)y^2$ is a multiple of $(a-x)$, of $2y$, of $4y^2$, &c.

ART. **108.**—A quantity that contains two or more quantities exactly, is a *common multiple* of them. Thus, 12 is a common multiple of 2 and 3; and $6ax$ is a common multiple of 2, 3, a, and x.

ART. **109.**—The *least common multiple* of two or more quantities, is the least quantity that will contain them exactly. Thus, 6 is the least common multiple of 2 and 3; and $10xy$ is the least common multiple of $2x$ and $5y$.

REMARK.—Two or more quantities can have but *one* least common multiple, while they may have an unlimited number of common multiples. Thus, while 6 is the least common multiple of 2 and 3, any multiple of 6, for instance, 12, 18, 24, &c., will be a common multiple of these numbers.

ART. **110.**—To find the least common multiple of two or more quantities.

It is evident, that one quantity will not contain another exactly, unless it contains the same prime factors. Thus, 30 does not exactly contain 14, because $30=2\times3\times5$, and $14=2\times7$; the prime factor 7, not being one of the prime factors of 30.

ART. **111.**—Any quantity will contain another exactly, if it contains all the prime factors of that quantity. Thus, 30 contains 6 exactly, because $30=2\times3\times5$, and $6=2\times3$; the prime factors 2 and 3 of the divisor, being also factors of the dividend. Hence, in order that one quantity shall contain another exactly, it is only necessary that it should contain all the prime factors of that quantity. Moreover, in order that any quantity shall exactly contain two or more quantities, it must contain all the different prime factors of those quantities. And, to be the least quantity that shall exactly contain them, it should contain these different prime factors only once, and no other factors besides. Hence, *the least common multiple of two or more quantities, contains all the different prime factors of these quantities once, and does not contain any other factor.*

Thus, the least common multiple of a^2bc and acx, is a^2bcx, since it contains all the factors in each of these quantities, and does not contain any other factor.

With this principle, let us find the least common multiple of ax, bx, and abc.

$$\begin{array}{c|ccc|}
a & ax & bx & abc \\ \cline{2-4}
x & x & bx & bc \\ \cline{2-4}
b & 1 & b & bc \\ \cline{2-4}
 & 1 & 1 & c
\end{array}$$

Arranging the quantities as in the margin, we see, that a is a factor common to two of the terms; hence it must be a factor of the least common multiple, and we place it on the left of the quantities. We then cancel this factor in each of the quantities in which it is found, which is done by dividing by it. By examining the remaining factors, it is seen that x is a common factor in the first and second terms. We then place it on the left, and cancel it in those terms in which it is

found. We next see, that b is a factor common to two of the quantities; hence, as before, we place it on the left, and cancel it in those terms in which it is found. We thus find, that a, x, b, and c, are all the prime factors in the given quantities; therefore, their product, $abcx$, will be the least common multiple of these quantities. Hence, the

RULE,

FOR FINDING THE LEAST COMMON MULTIPLE OF TWO OR MORE QUANTITIES.

1st. *Arrange the quantities in a horizontal line, and divide them by any prime factor that will divide two or more of them without a remainder, and set the quotients, together with the undivided quantities, in a line beneath.*

2d. *Continue dividing as before, until no prime factor, except unity, will divide two or more of the quantities, without a remainder.*

3d. *Multiply the divisors and the quantities in the last line together, and the product will be the least common multiple required.*

Or, *separate the given quantities into their prime factors, and then multiply together, such of those factors as are necessary to form a product that will contain all the prime factors in each quantity; this product will be the least common multiple required.*

ART. **112.**—Since the greatest common divisor of two quantities, contains all the factors common to them, it follows, that *if we divide the product of two quantities, by their greatest common divisor, the quotient will be their least common multiple.*

Find the least common multiple in each of the following examples.

1. $4a^2$, $3a^3x$, and $6ax^2y^3$. . . . . Ans. $12a^3x^2y^3$.
2. $12a^2x^2$, $6a^3$, and $8x^4y^2$. . . . . Ans. $24a^3x^4y^2$.
3. $6c^2nz^2$, $9n^4z$, and $12c^3n^2z^3$. . . . . Ans. $36c^3n^4z^3$.
4. 15, $6xz^2$, $9x^2z^4$, and $18cx^3$. . . . . Ans. $90cx^3z^4$.
5. $6a^4x^2y$, and $8a^2(a+x)$. . . . . Ans. $24a^4x^2y(a+x)$.
6. $4a^2(a-x)$, and $6ax^4(a^2-x^2)$. . . . . . Ans. $12a^2x^4(a^2-x^2)$.
7. $8x^2(x-y)$, $3a^4x^2$, and $12axy^2$. . . . . Ans. $24a^4x^2y^2(x-y)$.
8. $10a^2x^2(x-y)$, $15x^5(x+y)$, and $12(x^2-y^2)$. A. $60a^2x^5(x^2-y^2)$.

REVIEW.—107. What is a multiple of a quantity? Give an example. 108. What is a common multiple of two or more quantities? Give an example. 109. What is the least common multiple of two or more quantities? Give an example. How many common multiples may a quantity have? 110. When is one quantity not contained exactly in another? Give an example. 111. When is one quantity contained in another exactly? Give an example. What is necessary, in order that one quantity may exactly contain two or more quantities?

CHAPTER III.

ALGEBRAIC FRACTIONS.

DEFINITIONS AND FUNDAMENTAL PROPOSITIONS.

ART **113.**—IF a unit, or whole thing, is divided into any number of *equal* parts, one of the parts, or any number of them, is called a fraction.

Thus, if the line $A\,B$ be supposed to represent one foot, and be divided into four equal parts, one of those parts, as Ac, is called one fourth ($\frac{1}{4}$); two of them, as Ad, are called two fourths ($\frac{2}{4}$); and three of them, as Ae, are called three fourths ($\frac{3}{4}$).

In the algebraic fraction $\frac{1}{c}$, if $c=4$ and 1 denotes 1 foot, then $\frac{1}{c}$ denotes one fourth of a foot. In the fraction $\frac{a}{c}$, if $a=3$ and $\frac{1}{c}=\frac{1}{4}$ of a foot, then $\frac{a}{c}$ represents three fourths ($\frac{3}{4}$) of a foot.

ART. **114.**—Every quantity not expressed under the form of a fraction, is called an *entire* algebraic quantity. Thus, $ax+b$ is an entire quantity.

ART. **115.**—Every quantity composed partly of an entire quantity and partly of a fraction, is called a *mixed quantity.* Thus, $a+\frac{b}{x}$, is a mixed quantity.

ART. **116.**—An *improper algebraic fraction* is one whose numerator can be divided by the denominator, either with or without a remainder. Thus, $\frac{ab}{a}$, and $\frac{ax^2+b}{x}$, are improper fractions.

ART. **117.**—A single expression, as $\frac{1}{3}$, $\frac{a}{b}$, or $\frac{c}{d}$, is called a *simple fraction.* It may be either proper or improper.

REVIEW.—111. What is necessary, in order that any quantity may be the least, that shall contain two or more quantities exactly? What factors does the least common multiple of two or more quantities contain? What is the rule for finding the least common multiple of two or more quantities? How may the least common multiple of two or more quantities be found, by separating them into factors? 112. If the product of two quantities be divided by their greatest common divisor, what will the quotient be? 113. What is a fraction? 114. What is an entire algebraic quantity? Give an example. 115. What is a mixed quantity? Give an example. 116. What is an improper algebraic fraction? Give an example.

ART. **118.**—A fraction of a fraction, as $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{m}{n}$ of $\frac{a}{b}$, is called a *compound fraction.*

ART. **119.**—When a fraction has a fraction, either in its numerator, or in its denominator, or in both of them, it is called a *complex fraction.* Thus, $\dfrac{2\frac{1}{2}}{4}$, $\dfrac{3\frac{1}{2}}{2\frac{1}{3}}$, $\dfrac{a+\frac{b}{c}}{d}$, and $\dfrac{a+\frac{b}{c}}{e+\frac{m}{n}}$, are complex fractions.

ART. **120.**—Algebraic fractions are represented in the same manner as common fractions in Arithmetic. The number or quantity below the line, is called the *denominator*, because it *denominates*, or shows the number of parts into which the unit is divided; and the number or quantity above the line, is called the *numerator*, because it *numbers*, or shows how many parts are taken.

Thus, in the fraction, $\frac{3}{4}$, the denominator, 4, shows, that the unit (for instance, 1 foot,) is divided into 4 equal parts, and the numerator, 3, shows, that 3 of these parts are taken. Again, in the fraction $\frac{a}{c}$, the denominator c, shows, that a unit is divided into c equal parts, and a shows, that a of these parts are taken.

The numerator and denominator, are called the *terms* of a fraction.

ART. **121.**—In the preceding definitions of numerator and denominator, reference is had to a *unit* only. This is the simplest method of considering a fraction; but there is another point of view, in which it is proper to examine it.

If it be required to divide 3 apples equally, between 4 boys, it can be effected, by dividing each of the 3 apples into 4 equal parts, and then giving to each boy 3 of those parts, expressed by $\frac{3}{4}$. Now, the parts being equal to each other in size, it will be the same, for an individual to receive 3 parts from 1 apple, or 1 part from each of the 3 apples; that is, $\frac{3}{4}$ of one apple, is the same as $\frac{1}{4}$ of 3 apples; or, $\frac{3}{4}$ of 1 unit, is the same as $\frac{1}{4}$ of 3 units. Thus, $\frac{2}{5}$ may be regarded as expressing *two* fifths of *one* thing, or *one* fifth of *two* things.

REVIEW.—117. What is a simple fraction? Give an example. 118. What is a compound fraction? Give an example. 119. What is a complex fraction? Give an example. 120. In Algebraic Fractions, what is the quantity below the line called? Why? Above the line? Why? Give an example. What do you understand by the terms of a fraction?

So, $\frac{m}{n}$ is either the fraction $\frac{1}{n}$ of one unit taken m times, or it is the nth of m units. Hence, the numerator may be regarded, as showing the *number of units* to be divided; and the denominator, as showing the divisor, or *what part is taken* from each.

NOTE TO TEACHERS.—Although it is important that the pupil should be perfectly familiar with the principles contained in the following propositions, the demonstrations may be omitted, especially by the younger class of pupils, until the book is reviewed.

PROPOSITION I.

ART. **122.**—*If we multiply the numerator of a fraction, without changing the denominator, the value of the fraction is increased as many times as there are units in the multiplier.*

If we multiply the numerator of the fraction $\frac{2}{7}$ by 3, without changing the denominator, we get $\frac{6}{7}$. Thus:

$$\frac{2\times3}{7}=\frac{6}{7}$$

Now, $\frac{2}{7}$ and $\frac{6}{7}$ have the same denominator, and, therefore express parts of the same size; but the second fraction, $\frac{6}{7}$, has three times as large a numerator as the first, $\frac{2}{7}$; it therefore expresses three times as many of those equal parts as the first, and is, consequently, three times as large. And the same may be shown of any fraction whatever.

PROPOSITION II.

ART. **123.**—*If we divide the numerator of a fraction, without changing the denominator, the value of the fraction is diminished, as many times as there are units in the divisor.*

If we take the fraction $\frac{4}{5}$, and divide the numerator by 2, without changing the denominator, we get $\frac{2}{5}$. Thus:

$$\frac{4\div2}{5}=\frac{2}{5}$$

Now, $\frac{4}{5}$ and $\frac{2}{5}$ have the same denominator, and, therefore, express parts of the same size; but the numerator of the second fraction, $\frac{2}{5}$, is only one half as large as the numerator of the first, $\frac{4}{5}$; it therefore expresses only one half as many of those equal parts as the first, and is, consequently, only one half as large. And the same may be shown of other fractions.

REVIEW.—121. In what two different points of view may every fraction be regarded? Give examples. 122. How is the value of a fraction affected by multiplying the numerator only? How is this proposition proved? 123. How is the value of a fraction affected by dividing the numerator only? How is this proposition proved?

PROPOSITION III.

ART. **124.**—*If we multiply the denominator of a fraction, without changing the numerator, the value of the fraction is diminished, as many times as there are units in the multiplier.*

If we take the fraction $\frac{3}{4}$, and multiply the denominator by 2, without changing the numerator, we get $\frac{3}{8}$. Thus:

$$\frac{3}{4\times2}=\frac{3}{8}$$

Now, each of the fractions, $\frac{3}{4}$ and $\frac{3}{8}$, have the same numerator, and, therefore, express the same number of parts; but, in the second, the parts are only one half the size of those in the first; consequently, the whole value of the second fraction, is only *one half* that of the first. And the same may be shown of any fraction whatever.

PROPOSITION IV.

ART. **125.**—*If we divide the denominator of a fraction, without changing the numerator, the value of the fraction is increased as many times as there are units in the divisor.*

If we take the fraction $\frac{2}{9}$, and divide the denominator by 3 without changing the numerator, we get $\frac{2}{3}$. Thus:

$$\frac{2}{9\div3}=\frac{2}{3}$$

Now, each of the fractions, $\frac{2}{9}$ and $\frac{2}{3}$, have the same numerator, and, therefore, express the same number of parts; but, in the second, the parts are three times the size of those of the first; consequently, the whole value of the second fraction is three times that of the first. And the same may be shown of other fractions.

PROPOSITION V.

ART. **126.**—*Multiplying both terms of a fraction by the same number or quantity, changes the form of the fraction, but does not alter its value.*

If we multiply the numerator of a fraction by any number, its value (by Prop. I.) is *increased*, as many times as there are units in the multiplier; and, if we multiply the denominator, the value (by Prop. III.) is *decreased*, as many times as there are units in the multiplier. Hence, if both terms of a fraction are multiplied by the same number, the increase from multiplying the numerator,

REVIEW.—124. How is the value of a fraction affected by multiplying only the denominator? How is this proposition proved? 125. How is the value of a fraction affected by dividing the denominator only? How is this proposition proved? 126. How is the value of a fraction affected by multiplying both terms by the same quantity? Why?

is equal to the decrease from multiplying the denominator; consequently, the value remains unchanged.

PROPOSITION VI.

ART. **127.**—*Dividing both terms of a fraction by the same number or quantity, changes the form of the fraction, but does not alter its value.*

If we divide the numerator of a fraction by any number, its value (by Prop. II.) is *decreased*, as many times as there are units in the divisor; and if we divide the denominator, the value (by Prop. IV.) is *increased*, as many times as there are units in the divisor. Hence, if both terms of a fraction are divided by the same number, the decrease from dividing the numerator is equal to the increase from dividing the denominator; consequently, the value remains unchanged.

CASE I.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

ART. **128.**—Since the value of a fraction is not changed by dividing both terms by the same quantity (See Art. 127), we have the following

RULE.

Divide both terms by their greatest common divisor.

Or, *Resolve the numerator and denominator into their prime factors, and then cancel those factors common to both terms.*

REMARK.—The last rule will be found most convenient, when one or both terms are monomials.

1. Reduce $\frac{4ab^2}{6bx^2}$ to its lowest terms.

$$\frac{4ab^2}{6bx^2}=\frac{2ab\times2b}{3x^2\times2b}=\frac{2ab}{3x^2} \text{ Ans.}$$

Reduce the following fractions to their lowest terms.

2. $\frac{4a^3x^2}{6a^4}$. . . . . Ans. $\frac{2x^2}{3a}$.

3 $\frac{6a^2x^2}{8ax^3}$. . . . . Ans. $\frac{3a}{4x}$.

4. $\frac{6a^4x^2}{8a^2xy^4}$. . . Ans. $\frac{3a^2x}{4y^4}$.

5. $\frac{9x^4y^3z^5}{12x^3y^4z^5}$. . . . Ans. $\frac{3x}{4y}$.

6. $\frac{12x^2y^2z^4}{8x^2z^3}$. . Ans. $\frac{3y^2z}{2}$.

7. $\frac{8a^2b}{12ab^2+4abc}$. Ans. $\frac{2a}{3b+c}$.

8. $\frac{2a^2cx^2+2acx}{10ac^2x}$. Ans. $\frac{ax+1}{5c}$.

9. $\frac{5a^2b+5ab^2}{5abc+5abd}$. . Ans. $\frac{a+b}{c+d}$.

REVIEW.—127. How is the value of a fraction affected by dividing both terms by the same quantity? Why? 128. How do you reduce a fraction to its lowest terms?

10. $\frac{56x^2y}{24x^2y-40xy^2}$. . . . . Ans. $\frac{7x}{3x-5y}$.

11. $\frac{6ac}{12a^2c^2-18ac^2}$. . . . . Ans. $\frac{1}{2ac-3c}$.

12. $\frac{12x^2y-18xy^2}{18x^2y+12xy^2}$. . . . . Ans. $\frac{2x-3y}{3x+2y}$.

NOTE.—In the preceding examples, the greatest common divisor in each is a monomial; in those which follow, it is a polynomial; but, by separating the quantities into factors, or by the rule (Art. 106,) the greatest common divisor is readily found.

13. $\frac{3a^3-3ab^2}{5ab+5b^2}$. This is equal to

$$\frac{3a(a^2-b^2)}{5b(a+b)}=\frac{3a(a+b)(a-b)}{5b(a+b)}=\frac{3a(a-b)}{5b}.$$

14. $\frac{3z^3-24z+9}{4z^3-32z+12}$. Ans. $\frac{3}{4}$.

15. $\frac{5a^2+5ax}{a^2-x^2}$. Ans. $\frac{5a}{a-x}$.

16. $\frac{n^2-2n+1}{n^2-1}$. Ans. $\frac{n-1}{n+1}$.

17. $\frac{14a^2-7ab}{10ac-5bc}$. Ans. $\frac{7a}{5c}$.

18. $\frac{x^3-xy^2}{x^4-y^4}$. Ans. $\frac{x}{x^2+y^2}$.

19. $\frac{a^2+b^2}{a^4-b^4}$. Ans. $\frac{1}{a^2-b^2}$.

20. $\frac{x^2-y^2}{x^2-2xy+y^2}$. Ans. $\frac{x+y}{x-y}$.

21. $\frac{x^3-ax^2}{x^2-2ax+a^2}$. Ans. $\frac{x^2}{x-a}$.

22. $\frac{2x^2-6x}{x^2-x-6}$. Ans. $\frac{2x}{x+2}$.

23. $\frac{x^2+2x-15}{x^2+8x+15}$. Ans. $\frac{x-3}{x+3}$.

ART. **129.**—Exercises in Division (See Art. 76,) in which the quotient is a fraction, and capable of being reduced to lower terms.

1. Divide $5x^2y$ by $3xy^2$. . . . . Ans. $\frac{5x}{3y}$.

2. Divide $15a^2b^2c$ by $25a^3bc$. . . . . Ans. $\frac{3b}{5a}$.

3. Divide $25abc$ by $5ac^2$. . . . . Ans. $\frac{5b}{c}$.

4. Divide amn^2 by a^2m^2n. . . . . Ans. $\frac{n}{am}$.

In a similar manner, when one polynomial can not be exactly divided by another, the division may be indicated, and the result reduced to its most simple form.

5. Divide $25ax^2$ by $5ax^2-5axy$. . . . . Ans. $\frac{5x}{x-y}$.

6. Divide $3m^2+3n^2$ by $15m^2+15n^2$. . . . . Ans. $\frac{1}{5}$.

7. Divide $x^3y^2+x^2y^3$ by ax^2y+axy^2. . . . . Ans. $\frac{xy}{a}$.

8. Divide $4a+4b$ by $2a^2-2b^2$. . . . . Ans. $\frac{2}{a-b}$.

9. Divide n^3-2n^2 by n^2-4n+4. . . . . Ans. $\frac{n^2}{n-2}$

10. Divide x^2+2x-3 by x^2+5x+6. . . . . Ans. $\frac{x-1}{x+2}$.

CASE II.

TO REDUCE A FRACTION TO AN ENTIRE OR MIXED QUANTITY.

ART. **130.**—Since the numerator of the fraction may be regarded as a dividend, and the denominator as a divisor, this is merely a case of division. Hence, the

RULE.

Divide the numerator by the denominator, for the entire part, and, if there be a remainder, place it over the denominator for the fractional part.

NOTE.—The fractional part should be reduced to its lowest terms.

1. Reduce $\frac{3ax+b^2}{x}$ to a mixed quantity.

$$\frac{3ax+b^2}{x}=3a+\frac{b^2}{x}. \quad \text{Ans.}$$

Reduce the following fractions to entire or mixed quantities.

2. $\frac{ab+b^2}{a}$. . . . . Ans. $b+\frac{b^2}{a}$.

3. $\frac{cd-d^2}{d}$. . . . . Ans. $c-d$.

4. $\frac{a^2+x^2}{a-x}$. . . . . Ans. $a+x+\frac{2x^2}{a-x}$.

5. $\frac{2a^2x-x^3}{a}$. . . . . Ans. $2ax-\frac{x^3}{a}$.

6. $\frac{a^2-x^2+3}{a+x}$. . . . . Ans. $a-x+\frac{3}{a+x}$.

7. $\frac{4ax-2x^2-a^2}{2a-x}$. . . . . Ans. $2x-\frac{a^2}{2a-x}$.

8. $\frac{a^2-2ax}{a-x}$. . . . . Ans. $a-\frac{ax}{a-x}$,

9. $\frac{a^3+x^3-x^4}{a+x}$. . . . . Ans. $a^2-ax+x^2-\frac{x^4}{a+x}$,

10. $\frac{12x^3-3x^2}{4x^3-x^2-4x+1}$. . . . . Ans. $3+\frac{3}{x^2-1}$.

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CASE III.

TO REDUCE A MIXED QUANTITY TO THE FORM OF A FRACTION.

ART. **131.**—1. In $2\frac{1}{3}$ how many thirds?

In 1 unit there are 3 thirds; hence, in 2 units, there are twice as many, that is, 6; then, 6 thirds plus 1 third, are equal to 7 thirds; that is, $2\frac{1}{3}$ are equal to $\frac{7}{3}$. In the same manner, $a+\frac{b}{c}$ is equal to $\frac{ac}{c}+\frac{b}{c}$, which is equal to $\frac{ac+b}{c}$.

Hence, the

RULE,

FOR REDUCING A MIXED QUANTITY TO THE FORM OF A FRACTION.

Multiply the entire part by the denominator of the fraction; then add the numerator with its proper sign to the product, and place the result over the denominator.

REMARK.—Cases II. and III., are the reverse of, and mutually prove each other.

Before proceeding further, it is important for the learner to consider

THE SIGNS OF FRACTIONS.

ART. **132.**—It has been already stated (See Art. 121,) that in every fraction the numerator is a dividend, the denominator a divisor, and the value of the fraction the quotient. The signs prefixed to the terms of a fraction, affect only those terms; and the sign placed before a fraction, affects its whole value. Thus, in the fraction $-\frac{a^2-b^2}{x+y}$, the sign of a^2, the first term of the numerator, is plus; of the second, b^2, minus; while the sign of each term of the denominator, is plus. But the sign of the fraction, taken as a whole, is minus.

By the rule for the signs in Division, Art. 75, we have

$\frac{+ab}{+a}=+b$; or, changing the signs of both terms, $\frac{-ab}{-a}=+b$.

But, if we change the sign of the numerator, we have $\frac{-ab}{+a}=-b$.

And, if we change the sign of the denominator, we have $\frac{+ab}{-a}=-b$.

Hence, *the signs of both terms of a fraction may be changed, without altering its value, or changing its sign; but, if the sign of either term of a fraction be changed, and not that of the other, the sign of the fraction will be changed.*

From this, it also follows, that *the signs of either term of a fraction may be changed, without altering its value, if the sign of the fraction be changed at the same time.*

Thus, ... $\frac{ax-x^2}{c}=-\frac{ax-x^2}{-c}=-\frac{x^2-ax}{c}$.

And, . . . $a-\frac{a-x}{b}=a+\frac{a-x}{-b}=a+\frac{x-a}{b}$.

EXAMPLES.

1\. Reduce $3a+\frac{ax-a}{x}$ to a fractional form.

$$3a=\frac{3ax}{x} \text{ and } \frac{3ax}{x}+\frac{ax-a}{x}=\frac{3ax+ax-a}{x}=\frac{4ax-a}{x}. \quad \textbf{Ans.}$$

2\. Reduce $4a-\frac{a-b}{3c}$ to a fractional form.

$$4a=\frac{12ac}{3c} \text{ and } \frac{12ac}{3c}-\frac{a-b}{3c}=\frac{12ac-(a-b)}{3c}=\frac{12ac-a+b}{3c}. \quad \text{Ans.}$$

REMARK.—In solving this example, the learner should observe, that $\frac{a-b}{3c}$ is to be subtracted from $4a$. We reduce $4a$ to a quantity whose denominator is $3c$; then make the subtraction, and write the result over the common denominator, $3c$.

Reduce the following quantities to improper fractions.

3\. $5c+\frac{a-b}{2x}$. . . . . Ans. $\frac{10cx+a-b}{2x}$.

4\. $5c-\frac{a-b}{2x}$. . . . . Ans. $\frac{10cx-a+b}{2x}$.

5\. $3x+\frac{c-d}{xy}$. . . . . Ans. $\frac{3x^2y+c-d}{xy}$.

6\. $3x-\frac{4x^2-5}{5x}$. . . . . Ans. $\frac{11x^2+5}{5x}$.

7\. $8y+\frac{3a-y^2}{5y}$. . . . . Ans. $\frac{39y^2+3a}{5y}$.

8\. $x+y+\frac{x}{x+y}$. . . . . Ans. $\frac{x^2+2xy+y^2+x}{x+y}$.

9 $z-1+\frac{1-z}{1+z}$. . . . . Ans. $\frac{z^2-z}{z+1}$.

REVIEW.—130. How do you reduce a fraction to an entire or mixed quantity? 131. How do you reduce a mixed quantity to the form of a fraction? 132. What do the signs prefixed to the terms of a fraction affect? What does the sign placed before the whole fraction, affect? What effect does it have upon the value of a fraction, or upon its sign, to change the signs of both terms? To change the sign or signs of one term, and not of the other? To change the sign of the fraction, and one of its terms?

10. $\frac{4y}{2x+z}-5$. . . . . Ans. $\frac{4y-10x-5z}{2x+z}$

11. $\frac{3+5c}{8}-6$. . . . . Ans. $\frac{5c-45}{8}$

12. $3a^2x-\frac{a^2x^2-a^3}{x}$. . . . . Ans. $\frac{2a^2x^2+a^3}{x}$

13. $a+x+\frac{a^2+x^2}{a-x}$. . . . . Ans. $\frac{2a^2}{a-x}$.

14. $xy^2-\frac{x^2y^2-y^3}{x}$. . . . . Ans. $\frac{y^3}{x}$

15. $a^2-x^2-\frac{a^4+x^4}{a^2+x^2}$. . . . . Ans. $-\frac{2x^4}{a^2+x^2}$.

16. $a-x+\frac{a^2+x^2-5}{a+x}$. . . . . Ans. $\frac{2a^2-5}{a+x}$.

17. $a^3-a^2x+ax^2-x^3-\frac{a^4+x^4}{a+x}$. . . . . Ans. $-\frac{2x^4}{a+x}$.

CASE IV.

TO REDUCE FRACTIONS OF DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS, HAVING A COMMON DENOMINATOR.

ART. **133.**—1. Reduce $\frac{a}{b}$ and $\frac{c}{d}$ to a common denominator.

If we multiply both terms of the first fraction, $\frac{a}{b}$, by d, the denominator of the second, we shall have $\frac{a}{b}=\frac{a\times d}{b\times d}=\frac{ad}{bd}$; and, if we multiply both terms of the second fraction, $\frac{c}{d}$, by b, the denominator of the first, we shall have $\frac{c}{d}=\frac{c\times b}{d\times b}=\frac{bc}{bd}$.

In this solution we observe; *first*, the values of the fractions are not changed, since, in each fraction, both terms are multiplied by the same quantity; and, *second*, the denominators in each must be the same, since they consist of the product of the same quantities.

2. Reduce $\frac{a}{m}$, $\frac{b}{n}$, and $\frac{c}{r}$, to a common denominator.

Here, we are at liberty to multiply both terms of each fraction, by the same quantity, since this (See Art. 126) will not change its value. Now, if we multiply both terms of each fraction, by the denominators of the other two fractions, the new denominators in each will be the same, since, in each case, they will consist of the product of the same factors, that is, of all the denominators.

Thus . . . . $\frac{a\times n\times r}{m\times n\times r}=\frac{anr}{mnr}$.

$$\frac{b\times m\times r}{n\times m\times r}=\frac{bmr}{mnr}.$$

$$\frac{c\times m\times n}{r\times m\times n}=\frac{cmn}{mnr}.$$

It is evident, that the value of each fraction is not changed, and that they have the same denominators. Hence, the

RULE,

FOR REDUCING FRACTIONS TO A COMMON DENOMINATOR.

Multiply both terms of each fraction by the product of all the denominators, except its own.

REMARK.—Since each denominator of the new fractions, will consist of the product of all the denominators of the given fractions, it is unnecessary to perform the same multiplication more than once.

EXAMPLES.

Reduce the following fractions, in each example, to others, having a common denominator.

3. $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{1}{2}$. . . . . Ans. $\frac{2ad}{2bd}$, $\frac{2bc}{2bd}$, and $\frac{bd}{2bd}$.

4. $\frac{x}{y}$, and $\frac{x+a}{c}$. . . . . Ans. $\frac{cx}{cy}$, and $\frac{xy+ay}{cy}$.

5. $\frac{2}{3}$, $\frac{3a}{4}$, and $\frac{x-y}{b}$. . . . . . Ans. $\frac{8b}{12b}$, $\frac{9ab}{12b}$, and $\frac{12x-12y}{12b}$.

6. $\frac{2x}{3y}$, $\frac{3x}{5z}$, and a. . . . . Ans. $\frac{10xz}{15yz}$, $\frac{9xy}{15yz}$, and $\frac{15ayz}{15yz}$.

7. $\frac{a}{x}$, $\frac{x}{y}$, and $\frac{y}{z}$. . . . . Ans. $\frac{ayz}{xyz}$, $\frac{x^2z}{xyz}$, and $\frac{xy^2}{xyz}$.

8. $\frac{1}{2}$, $\frac{x^2}{3}$, and $\frac{x^2+z^2}{x+z}$. . . Ans. $\frac{3x+3z}{6x+6z}$, $\frac{2x^3+2x^2z}{6x+6z}$, and $\frac{6x^2+6z^2}{6x+6z}$.

9 $\frac{x+y}{x-y}$, and $\frac{x-y}{x+y}$. . . . . Ans. $\frac{x^2+2xy+y^2}{x^2-y^2}$, and $\frac{x^2-2xy+y^2}{x^2-y^2}$.

10. a, $\frac{3b}{c}$, d, and 5. . . . . Ans. $\frac{ac}{c}$, $\frac{3b}{c}$, $\frac{cd}{c}$, and $\frac{5c}{c}$.

REVIEW.—133. How do you reduce fractions of different denominators to equivalent fractions having the same denominator? Why is the value of each fraction not changed by this process? Why does this process give to each fraction the same denominator?

11. $\frac{a}{3m}$, $\frac{m-n}{a}$, and $-\frac{a}{m+n}$.

Ans. $\frac{a^2m+a^2n}{3am^2+3amn}$, $\frac{3m^3-3mn^2}{3am^2+3amn}$, and $\frac{3a^2m}{3am^2+3amn}$.

ART. **134.**—It frequently happens, that the denominators of the fractions to be reduced, contain one or more common factors. In such cases, the preceding rule does not give the *least* common denominator. From the preceding Article we see, that the common denominator is a multiple of all the denominators; and, that each numerator is multiplied by a quantity which is equal to the quotient obtained, by dividing this multiple by its denominator. Thus, in the second example, nr, mr, and mn, the quantities by which each numerator is respectively multiplied, may be regarded as the quotients obtained, by dividing mnr successively, by m, n, and r. Now, if we obtain the least common multiple of the denominators, by the rule, Case III., and then divide it by each denominator respectively, and multiply the quotients by the numerators respectively, we shall obtain a new class of fractions, equivalent to the former, and having for a common denominator, the least common multiple of the given denominators. It is easily seen, that both terms of each fraction are multiplied by the same quantity, and hence, that the resulting fractions are equivalent to the given ones.

1. Reduce $\frac{m}{b}$, $\frac{n}{bc}$, and $\frac{r}{cd}$, to equivalent fractions, having the least common denominator.

The least common multiple of the denominators is easily found to be bcd; dividing this by b, the denominator of the first fraction, the quotient is cd; then multiplying both terms of $\frac{m}{b}$ by cd, the result is . . . . $\frac{mcd}{bcd}$.

Then $bcd \div bc = d$, and $\frac{n \times d}{bc \times d} =$ . . . . . $\frac{nd}{bcd}$.

Also, $bcd \div cd = b$, and $\frac{r \times b}{cd \times b} =$ . . . . . $\frac{br}{bcd}$.

The process of multiplying the denominators by the quotients may be omitted, as the product in each case will be equal to the least common multiple. Hence, the

RULE,

FOR REDUCING FRACTIONS OF DIFFERENT DENOMINATORS, TO EQUIVALENT FRACTIONS, HAVING THE LEAST COMMON DENOMINATOR.

1st. *Find the least common multiple of all the denominators; this will be the common denominator.*

2d. *Divide the least common multiple, by the first of the given denominators, and multiply the quotient by the first of the given numerators; the product will be the first of the required numerators.*

3d. *Proceed, in a similar manner, to find each of the other numerators.*

NOTE.—Each fraction should be in its lowest terms, before commencing the operation.

Reduce the following fractions, in each example, to equivalent fractions, having the least common denominator.

2. $\frac{2a}{3bc}$, $\frac{3x}{cd}$, and $\frac{5y}{6bd}$. . . . . . . Ans. $\frac{4ad}{6bcd}$, $\frac{18bx}{6bcd}$, and $\frac{5cy}{6bcd}$.

3. $\frac{m}{ac}$, $\frac{n}{b^2c}$, and $\frac{r}{c^2d}$. . . . . Ans. $\frac{b^2cdm}{ab^2c^2d}$, $\frac{acdn}{ab^2c^2d}$, and $\frac{ab^2r}{ab^2c^2d}$.

4. $\frac{x+y}{x-y}$, $\frac{x-y}{x+y}$, and $\frac{x^2+y^2}{x^2-y^2}$. . . Ans. $\frac{(x+y)^2}{x^2-y^2}$, $\frac{(x-y)^2}{x^2-y^2}$, and $\frac{x^2+y^2}{x^2-y^2}$.

Other exercises will be found in the addition of fractions.

NOTE.—The two following Articles depend on the same principle as the two preceding, and are, therefore, introduced here. They will both be found of frequent use, particularly in completing the square, in the solution of equations of the second degree.

ART. **135.**—To reduce an entire quantity to the form of a fraction having a given denominator.

1. Let it be required to reduce a to a fraction having b for its denominator.

Since any quantity may be reduced to the form of a fraction, by writing 1 beneath it, a is the same as $\frac{a}{1}$; if we multiply both terms by b, which will not change its value (See Art. 126), we have $\frac{a}{1}=\frac{ab}{b}$, for the required fraction. Hence, the

RULE,

FOR REDUCING AN ENTIRE QUANTITY TO THE FORM OF A FRACTION HAVING A GIVEN DENOMINATOR.

Multiply the entire quantity by the given denominator, and write the product over it.

EXAMPLES.

2. Reduce x to a fraction, whose denominator is 4. Ans. $\frac{4x}{4}$.

3. Reduce m to a fraction, whose denominator is $9a^2$.

Ans. $\frac{9a^2m}{9a^2}$.

REVIEW.—134. How do you reduce fractions of different denominators to equivalent fractions, having the least common denominator?

4. Reduce $3c+5$ to a fraction whose denominator is $16c^2$.

Ans. $\frac{48c^3+80c^2}{16c^2}$.

5. Reduce $a-b$ to a fraction, whose denominator is $a^2-2ab+b^2$.

Ans. $\frac{a^3-3a^2b+3ab^2-b^3}{a^2-2ab+b^2}=\frac{(a-b)^3}{(a-b)^2}$.

ART. **136.**—To convert a fraction to an equivalent one, having a denominator equal to some multiple of the denominator of the given fraction.

1. Reduce $\frac{a}{b}$ to a fraction, whose denominator is bc.

It is evident, that the terms must be multiplied by the same quantity, so as not to change the value of the fraction. It is then required to find, what the denominator, b, must be multiplied by, that the product shall become bc; but, it is evident, this multiple will be found, by dividing bc by b, which gives the quotient, c. Then, multiplying both terms of the fraction $\frac{a}{b}$ by c, the result is $\frac{ac}{bc}$, which is equal to the given fraction $\frac{a}{b}$, and has, for its denominator bc. Hence, the

RULE,

FOR CONVERTING A FRACTION TO AN EQUIVALENT ONE, HAVING A GIVEN DENOMINATOR.

Divide the given denominator by the denominator of the given fraction, and multiply both terms by the quotient.

REMARK.—This rule is perfectly general, but it is never applied, except where the required denominator is a multiple of the given one. In other cases, it would produce a complex fraction. Thus, if it is required to convert $\frac{1}{2}$ into an equivalent fraction, whose denominator is 5, the numerator of the new fraction would be $2\frac{1}{2}$.

2. Convert $\frac{3}{4}$ to an equivalent fraction, having the denominator 16. Ans. $\frac{12}{16}$.

3. Convert $\frac{a}{3}$ to an equivalent fraction, having the denominator 9. Ans. $\frac{3a}{9}$.

4. Convert $\frac{b}{c}$ to an equivalent fraction, having the denominator a^2c^2. Ans. $\frac{a^2bc}{a^2c^2}$.

REVIEW.—134. If each fraction is not in its lowest terms, before commencing the operation, what is to be done? 135. How do you reduce an entire quantity to the form of a fraction having a given denominator?

5. Convert $\frac{m+n}{m-n}$ to an equivalent fraction, having the denominator $m^2-2mn+n^2$. Ans. $\frac{m^2-n^2}{m^2-2mn+n^2}$.

6. Convert $\frac{a}{b+c}$ to an equivalent fraction, having the denominator $a^2(b+c)^2$. Ans. $\frac{a^3(b+c)}{a^2(b+c)^2}$.

CASE V.

ADDITION AND SUBTRACTION OF FRACTIONS.

ART. **137.**—1. Let it be required to find the sum of $\frac{2}{5}$ and $\frac{4}{5}$.

Here, both parts being of the same kind, that is, fifths, we may add them together, and the sum is 6 fifths, ($\frac{6}{5}$).

2. Let it be required to find the sum of $\frac{a}{m}$ and $\frac{b}{m}$.

Here, the parts being of the same kind, that is, $m$ths, we may, as in the first case, add the numerators, and write the result over the common denominator.

Thus, ... $\frac{a}{m}+\frac{b}{m}=\frac{a+b}{m}$.

3. Again, let it be required to find the sum of $\frac{a}{m}$ and $\frac{c}{n}$.

Here, the parts not being of the same kind, that is, the denominators being different, we can not add the numerators together, and call them by the same name. We may, however, reduce them to a common denominator, and then add them together.

Thus, $\frac{a}{m}=\frac{an}{mn}$; $\frac{c}{n}=\frac{cm}{mn}$. And $\frac{an}{mn}+\frac{cm}{mn}=\frac{an+cm}{mn}$.

Hence, the

RULE,

FOR THE ADDITION OF FRACTIONS.

Reduce the fractions, if necessary, to a common denominator; add the numerators together, and place their sum over the common denominator.

ART. **138.**—It is obvious, that the same principles would apply, if it were required to find the difference between two fractions; that is, if their denominators were the same, the numerators might be subtracted; but, if their denominators were different, it would be necessary to reduce them to the same denominator, before performing the subtraction. Hence, the

RULE,

FOR THE SUBTRACTION OF FRACTIONS.

Reduce the fractions, if necessary, to a common denominator; then subtract the numerator of the fraction to be subtracted from the numerator of the other, and place the remainder over the common denominator.

EXAMPLES IN ADDITION OF FRACTIONS.

4. Add $\frac{a}{2}$, $\frac{a}{3}$, and $\frac{a}{6}$, together. . . . . Ans. a.

5. Add $\frac{x}{3}$, $\frac{x}{5}$, and $\frac{x}{6}$ together. . . . . Ans. $\frac{7x}{10}$.

6. Add $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ together. . . . . Ans. $\frac{bc+ac+ab}{abc}$.

7. Add $\frac{x}{2}$, $\frac{y}{3}$, and $\frac{z}{4}$ together. . . . . Ans. $\frac{6x+4y+3z}{12}$.

8. Add $\frac{3x}{4}$, $\frac{4x}{5}$, and $\frac{5x}{6}$ together. . . Ans. $\frac{143x}{60}=2x+\frac{23x}{60}$.

9. Add $\frac{x+y}{2}$ and $\frac{x-y}{2}$ together. . . . . Ans. x.

10. Add $\frac{1}{a+b}$ and $\frac{1}{a-b}$ together. . . . . Ans. $\frac{2a}{a^2-b^2}$.

11. Add $\frac{x}{x+y}$ and $\frac{y}{x-y}$ together. . . . . Ans. $\frac{x^2+y^2}{x^2-y^2}$.

12. Add $\frac{5+x}{y}$, $\frac{3-ax}{ay}$, and $\frac{b}{3a}$ together. . . Ans. $\frac{15a+by+9}{3ay}$.

13. Add $\frac{a-b}{ab}$, $\frac{b-c}{bc}$, and $\frac{c-a}{ac}$ together. . . . . . Ans. 0.

14. Add $\frac{1}{1+x}$, $\frac{x}{1-x}$, and $\frac{x}{1+x}$ together. . . . . . Ans. $\frac{1}{1-x}$.

When entire quantities and fractions are to be added together, they may be connected by the sign of addition, or the entire quantities and the fractions may be reduced to a common denominator, and the addition then performed.

15. Add $2x$, $3x+\frac{3z}{5}$, and $x+\frac{2z}{9}$ together. . . Ans. $6x+\frac{37z}{45}$.

16. Add $5x+\frac{x-2}{3}$ and $4x-\frac{2x-3}{5x}$ together.

Ans. $9x+\frac{5x^2-16x+9}{15x}$.

17. Add $3+\frac{2a}{x}$, $5-\frac{3a-2x}{x}$, and $7+\frac{x-a}{a}$ together.

Ans. $15-\frac{a^2-ax-x^2}{ax}$.

18. Add $\frac{a+b}{a-b}$, $\frac{a-b}{a+b}$, and 2 together. . . . . Ans. $\frac{4a^2}{a^2-b^2}$

REVIEW.—136. How do you convert a fraction to an equivalent one, having a given denominator? Explain the operation by an example. 137. When fractions have the same denominator, how do you add them together? When fractions have different denominators, how do you add them together?

EXAMPLES IN SUBTRACTION OF FRACTIONS.

1. From $\frac{a}{2}$ take $\frac{a}{3}$. . . . . Ans. $\frac{a}{6}$

2. From $\frac{3x}{4}$ take $\frac{2x}{3}$. . . . . Ans. $\frac{x}{12}$.

3. From $\frac{a+b}{2}$ take $\frac{a-b}{2}$. . . . . Ans. b.

4. From $\frac{2ax}{3}$ take $\frac{5ax}{2}$. . . . . Ans. $-\frac{11ax}{6}$.

5. From $\frac{3}{4a}$ take $\frac{5}{2x}$. . . . . Ans. $\frac{3x-10a}{4ax}$.

6. From $\frac{3a}{4x}$ take $\frac{4x}{3a}$. . . . . Ans. $\frac{9a^2-16x^2}{12ax}$.

7. From $\frac{x+y}{x-y}$ take $\frac{x-y}{x+y}$. . . . . Ans. $\frac{4xy}{x^2-y^2}$.

8. From $\frac{a^2+ax}{x-y}$ take $\frac{a^2-ax}{x+y}$. . . . . Ans. $\frac{2ax^2+2a^2y}{x^2-y^2}$.

9. From $\frac{2a+b}{5c}$ take $\frac{3a-b}{7c}$. . . . . Ans. $\frac{12b-a}{35c}$.

10. From $5x+\frac{x}{b}$ take $2x-\frac{x-b}{c}$. . . . Ans. $3x+\frac{bx+cx-b^2}{bc}$.

11. From $\frac{1}{a-b}$ take $\frac{1}{a+b}$. . . . . Ans. $\frac{2b}{a^2-b^2}$.

12. From $a+b$ take $\frac{1}{a}+\frac{1}{b}$. . . . . Ans. $\frac{a^2b+ab^2-a-b}{ab}$.

13. From $\frac{x^3+y^3}{x-y}$ take $\frac{x^3-y^3}{x+y}$. . . . . Ans. $\frac{2x^3y+2xy^3}{x^2-y^2}$.

14. From $\frac{1+a^2}{1-a^2}$ take $\frac{1-a^2}{1+a^2}$. . . . . Ans. $\frac{4a^2}{1-a^4}$.

15. From $\frac{2(a^2+b^2)}{a^2-b^2}$ take $\frac{a-b}{a+b}$. . . . . Ans. $\frac{a+b}{a-b}$.

16. From $\frac{1}{x+1}$ take $\frac{x-2}{x^2-x+1}$. . . . . Ans. $\frac{3}{1+x^3}$.

17. From $x+\frac{1}{x-1}$ take $\frac{2}{x+1}$. . . . . Ans. $\frac{x^3-2x+3}{x^2-1}$.

18. From $2a-3x+\frac{a-x}{a}$ take $a-5x+\frac{x-a}{x}$. A. $a+2x+\frac{a^2-x^2}{ax}$.

19. From $a+x+\frac{x}{x^2-y^2}$ take $a-x+\frac{1}{x+y}$. . Ans. $2x+\frac{y}{x^2-y^2}$.

REVIEW.—138. If two fractions have the same denominator, how do you find their difference? When two fractions have different denominators, how do you find their difference?

CASE VI.

TO MULTIPLY ONE FRACTIONAL QUANTITY BY ANOTHER.

ART. **139.**—To multiply a fraction by an entire quantity, or an entire quantity by a fraction.

It is evident, from Prop. I., Art. 122, that in multiplying the numerator of a fraction by an entire quantity, the fraction is increased as many times as there are units in the multiplier. Thus, $\frac{a}{b}$ taken twice, is $\frac{2a}{b}$; and taken m times, is $\frac{ma}{b}$.

Again, when two quantities are to be multiplied together, either may be made the multiplier (Art. 67); to multiply 4 by $\frac{2}{3}$, is the same as to multiply $\frac{2}{3}$ by 4. Or, to multiply m by $\frac{a}{b}$, is the same as to multiply $\frac{a}{b}$ by m. Hence, the

RULE,

FOR THE MULTIPLICATION OF A FRACTION BY AN ENTIRE QUANTITY, OR OF AN ENTIRE QUANTITY BY A FRACTION.

Multiply the numerator by the entire quantity, and write the product over the denominator.

Since (See Art. 125,) dividing the denominator of a fraction increases the value of the fraction, as many times as there are units in the divisor, it is evident, that any fraction will be multiplied by an entire quantity, if the denominator of the fraction be divided by the entire quantity. Thus, in multiplying $\frac{5}{8}$ by 2, we may divide the denominator by 2, and the result will be $\frac{5}{4}$, which is the same as to multiply by 2, and reduce the resulting fraction to its lowest terms. Hence, *in multiplying a fraction and an entire quantity together, we should always divide the denominator of the fraction by the entire quantity, when it can be done without a remainder.*

REMARK.—The expression, "What is two thirds of 6?" has the same meaning, as "What is the product of 6 multiplied by $\frac{2}{3}$?" The reason of the rule for the multiplication of an entire quantity by a fraction, may be shown otherwise, thus: one third of a is $\frac{a}{3}$; two thirds is twice as much as one third, that is, two thirds of a is $\frac{2a}{3}$. Also, $\frac{1}{n}$ of a is $\frac{a}{n}$, and the $\frac{m}{n}$ part of a is $\frac{ma}{n}$.

REVIEW.—139. How do you multiply a fraction by an entire quantity, or an entire quantity by a fraction? When the denominator of the fraction is a multiple of the entire quantity, what is the shortest method of finding their product?

EXAMPLES.

1. Multiply $\frac{2a}{bc}$ by ad. . . . . Ans. $\frac{2a^2d}{bc}$.

2. Multiply $\frac{a+b}{c}$ by xy. . . . . Ans. $\frac{axy+bxy}{c}$.

3. Multiply $\frac{b+c}{d}$ by $b-c$. . . . . Ans. $\frac{b^2-c^2}{d}$.

4. Multiply $\frac{3x^2}{10y}$ by $5y$. . . . . Ans. $\frac{3x^2}{2}$.

5. Multiply $a-2b$ by $\frac{4c}{2a+c}$. . . . . Ans. $\frac{4ac-8bc}{2a+c}$.

6. Multiply a^2-b^2 by $\frac{3c-a}{2a}$. . . Ans. $\frac{3a^2c-3b^2c-a^3+ab^2}{2a}$.

7. Multiply $\frac{b+c}{b-c}$ by $a+c$ . . . . Ans. $\frac{ab+ac+bc+c^2}{b-c}$.

8. Multiply $\frac{2a+b}{a^2-b^2}$ by $a-b$. . . . . Ans. $\frac{2a+b}{a+b}$.

9. Multiply $\frac{2a+3xz}{a^2b}$ by ab. . . . . Ans. $\frac{2a+3xz}{a}$.

10. Multiply $\frac{5bc+3bx}{10x^3y^2-14x^4y^3}$ by $2x^2y^2$. . . . Ans. $\frac{5bc+3bx}{5x-7x^2y}$.

11. Multiply $\frac{abc}{3(x^4-y^4)}$ by x^2+y^2. . . . . . Ans. $\frac{abc}{3(x^2-y^2)}$.

12. Multiply $\frac{ax+by}{4(a+b)(a-b)}$ by $2(a-b)$. . . . Ans. $\frac{ax+by}{2(a+b)}$.

13. Multiply $\frac{5c+4d}{5(a-b)(c^2-d^2)}$ by $5(a-b)(c+d)$. Ans. $\frac{5c+4d}{c-d}$.

14. Multiply $\frac{a}{c}$ by c. . . . . Ans. $\frac{ac}{c}=a$, or $\frac{a}{1}$.

Hence, we see, that *if a fraction is multiplied by a quantity equal to its denominator, the product will be equal to the numerator.*

15. Multiply $\frac{a-b}{c+d}$ by $c+d$. . . . . Ans. $a-b$.

16. Multiply $\frac{m^2-n^2}{2x+5y}$ by $2x+5y$. . . . . . . Ans. m^2-n^2

Art. 140.—To multiply a fraction by a fraction.

1. Let it be required to find the product of $\frac{4}{5}$ multiplied by $\frac{2}{3}$. Since $\frac{2}{3}$ is the same as 2 multiplied by $\frac{1}{3}$, it is required to multiply $\frac{4}{5}$ by 2, and take $\frac{1}{3}$ of the product. Now, $\frac{4}{5}$ multiplied by 2, is equal to $\frac{8}{5}$, and $\frac{1}{3}$ of $\frac{8}{5}$, is equal to $\frac{8}{15}$ (since, to take $\frac{1}{3}$ is to divide by 3, and any fraction is divided, by multiplying its denominator, by Art. 124.) Hence, the product of $\frac{4}{5}$ and $\frac{2}{3}$ is $\frac{8}{15}$.

In the same manner, if it were required to multiply $\frac{a}{c}$ by $\frac{m}{n}$; since $\frac{m}{n}=m\times\frac{1}{n}$, we would multiply $\frac{a}{c}$ by m, and take $\frac{1}{n}$ of the product. Thus, $\frac{a}{c}\times m=\frac{ma}{c}$, and $\frac{1}{n}$ of $\frac{ma}{c}=\frac{ma}{nc}$. Hence, the

RULE,

FOR THE MULTIPLICATION OF A FRACTION BY A FRACTION.

Multiply the numerators together, for a new numerator; and the denominators together, for a new denominator.

REMARKS.—1st. If either of the factors is a mixed quantity, it is best to reduce it to an improper fraction, before commencing the operation.

2d. The expression, "What is one third of one fourth," has the same meaning as "What is the product of $\frac{1}{3}$ multiplied by $\frac{1}{4}$." Also, the expression, "What is two thirds of three fourths," has the same meaning as "What is the product of $\frac{2}{3}$ multiplied by $\frac{3}{4}$."

3d. When the numerators and denominators have common factors, it is best to indicate the multiplication, and then cancel the factors common to both terms, after which, the remaining terms may be multiplied together.

Thus, $\frac{14a}{15b}\times\frac{5c}{21d}=\frac{2\times\cancel{7}\times\cancel{5}ac}{\cancel{5}\times3\times3\times\cancel{7}bd}=\frac{2ac}{9bd}$.

Also, $\frac{5a}{a^2-b^2}\times\frac{a+b}{2a}=\frac{5a(a+b)}{2a(a+b)(a-b)}=\frac{5}{2(a-b)}$.

EXAMPLES.

1. Multiply $\frac{3a}{4}$ by $\frac{5x}{8}$. . . . . Ans. $\frac{15ax}{32}$.
2. Multiply $\frac{4a}{5x}$ by $\frac{3x}{7a}$. . . . . Ans. $\frac{12}{35}$.
3. Multiply $\frac{2a}{3}$ by $\frac{4a}{5}$. . . . . Ans. $\frac{8a^2}{15}$.
4. Multiply $\frac{3x^2}{10y}$ by $\frac{5y}{9x}$. . . . . Ans. $\frac{x}{6}$.
5. Multiply $\frac{3(a+x)}{2}$ by $\frac{4x}{a+x}$. . . . . Ans. $6x$.
6. Multiply $\frac{2x+3}{5}$ by $\frac{10x}{7}$. . . . . Ans. $\frac{4x^2+6x}{7}$.

REVIEW.—140. How do you multiply one fraction by another? Explain the reason of the rule, by analyzing an example. When one of the factors is a mixed quantity, what ought to be done? What is the meaning of the expression, "What is one third of one fourth?" How may the work be shortened, when the numerator and denominator have common factors?

7. Multiply $\frac{x^2-y^2}{ab}$ by $\frac{a^2}{x+y}$. . . . . Ans. $\frac{a(x-y)}{b}$.

8. Multiply $\frac{xyz}{x^4+y^3}$ by $\frac{x^4+y^3}{xyz}$. . . . . Ans. 1.

9. Multiply $\frac{a}{a-b}$ by $\frac{b}{a+b}$. . . . . Ans. $\frac{ab}{a^2-b^2}$.

10. Multiply $\frac{a-x}{x^2}$ by $\frac{a^2}{a^2-x^2}$. . . . . Ans. $\frac{a^2}{x^2(a+x)}$.

11. Multiply $\frac{x}{a+x}$, $\frac{a^2-x^2}{x^2}$, and $\frac{a}{a-x}$, together. . . . Ans. $\frac{a}{x}$.

12. Multiply $\frac{x^2+y^2}{x-y}$, $\frac{x^2-y^2}{x+y}$, and a together. . Ans. $a(x^2+y^2)$.

13. Multiply $\frac{a-b}{2}$, $\frac{2}{a^2-b^2}$, and $a+b$ together. . . . Ans. 1.

14. Multiply $\frac{x^2+y^2}{x^2-y^2}$ by $\frac{x-y}{x+y}$. . . . . Ans. $\frac{x^2+y^2}{x^2+2xy+y^2}$.

15. Multiply $\frac{a-b}{5}$ by $\frac{15x+25}{a^2-b^2}$. . . . . Ans. $\frac{3x+5}{a+b}$.

16. Multiply $c+\frac{cx}{c-x}$ by $\frac{c^2-x^2}{x+1}$. . . . . Ans. $\frac{c^2(c+x)}{x+1}$.

CASE VII.

TO DIVIDE ONE FRACTIONAL QUANTITY BY ANOTHER.

ART. **141.**—To divide a fraction by an entire quantity.

It has been shown, in Art. 123 and Art. 124, that a fraction is divided by an entire quantity, by dividing its numerator, or multiplying its denominator. Thus, $\frac{4}{5}$ divided by 2, or $\frac{1}{2}$ of $\frac{4}{5}$, is $\frac{2}{5}$. $\frac{3a}{b}$ divided by 3, or $\frac{1}{3}$ of $\frac{3a}{b}$, is $\frac{a}{b}$. $\frac{ma}{n}$ divided by m, or $\frac{1}{m}$ of $\frac{ma}{n}$, is $\frac{a}{n}$.

Or, by multiplying the denominator; $\frac{4}{5}$ divided by 2, is equal to $\frac{4}{10}$, since the number of parts in the numerator is the same, but only half as large as before, $\frac{1}{10}$ being the half of $\frac{1}{5}$. Hence, the

RULE,

FOR DIVIDING A FRACTION BY AN ENTIRE QUANTITY.

Divide the numerator by the divisor, if it can be done without a remainder; if not, multiply the denominator by the entire quantity, and write the numerator over the result.

NOTE.—If the numerator of the fraction and the entire quantity, contain common factors, it is best to indicate the operation, and cancel the common factors; the result found thus will be in its lowest terms.

The preceding rule may be derived in another manner, thus: To divide a number by 2, is to take $\frac{1}{2}$ of it, or to multiply it by $\frac{1}{2}$; to divide by 3, is to take $\frac{1}{3}$ of it, or to multiply it by $\frac{1}{3}$. In the same manner, to divide a quantity by m, is to take $\frac{1}{m}$ of it, or to multiply it by $\frac{1}{m}$. Hence, *to divide a fraction by an entire quantity, we write the divisor in the form of a fraction* (thus, $m=\frac{m}{1}$), *and invert it, and then proceed as in multiplication of fractions.*

EXAMPLES.

1. Divide $\frac{6a^2b}{7n}$ by $3ab$. . . . . Ans. $\frac{2a}{7n}$.
2. Divide $\frac{15a^3c^2}{17bd}$ by $5a^2c$. . . . . Ans. $\frac{3ac}{17bd}$.
3. Divide $\frac{14ac^3m^2}{11xy}$ by $7acm^2$. . . . . Ans. $\frac{2c^2}{11xy}$.
4. Divide $\frac{35b^3d^2}{13ac^2}$ by $5b^3d$. . . . . Ans. $\frac{7d}{13ac^2}$.
5. Divide $\frac{a^2+ab}{3+2x}$ by a. . . . . Ans. $\frac{a+b}{3+2x}$.
6. Divide $\frac{c^2+cd}{5}$ by $c+d$. . . . . Ans. $\frac{c}{5}$.
7. Divide $\frac{x^2+2xy+y^2}{c+d}$ by $x+y$. . . . . Ans. $\frac{x+y}{c+d}$.
8. Divide $\frac{a^3-b^3}{2b+3c}$ by $a-b$. . . . . Ans. $\frac{a^2+ab+b^2}{2b+3c}$.
9. Divide $\frac{6a^3+7a^2-5a}{c-g}$ by $3a+5$. . . . . . Ans. $\frac{2a^2-a}{c-g}$.
10. Divide $\frac{8x^3-8x+3}{ab+cd^2}$ by $4x^2+2x-3$. . . . . Ans. $\frac{2x-1}{ab+cd^2}$.
11. Divide $\frac{2a}{3c}$ by b. . . . . Ans. $\frac{2a}{3bc}$.
12. Divide $\frac{3}{ab+cd}$ by bd. . . . . Ans. $\frac{3}{ab^2d+bcd^2}$.
13 Divide $\frac{3+5a}{a-b}$ by $a+b$. . . . . Ans. $\frac{3+5a}{a^2-b^2}$.

REVIEW.—141. How do you divide a fraction by an entire quantity? Explain the reason of the rule, by analyzing an example. How may the work be abbreviated, when the numerators of the fraction and the entire quantity contain common factors?

14 Divide $\frac{2x+3y}{4x-3y}$ by xy. . . . . Ans. $\frac{2x+3y}{4x^2y-3xy^2}$.

15. Divide $\frac{3a+5c}{2x+3y}$ by $2x-3y$. . . . . Ans. $\frac{3a+5c}{4x^2-9y^2}$.

16. Divide $\frac{b-c}{a^2+ab+b^2}$ by $a-b$. . . . . Ans. $\frac{b-c}{a^3-b^3}$.

17. Divide $\frac{x-y}{x^2-xy+y^2}$ by $x+y$. . . . . Ans. $\frac{x-y}{x^3+y^3}$.

18. Divide $\frac{a}{a^2-ax+x^2}$ by a^2+ax+x^2. . . . Ans. $\frac{a}{a^4+a^2x^2+x^4}$.

19. Divide $\frac{a^2+abc}{b+c}$ by $a+bc$. . . . . Ans. $\frac{a}{b+c}$

20. Divide $\frac{15a^2b+3ac}{a+b}$ by $3a^2b$. . . . . . . Ans. $\frac{5ab+c}{a^2b+ab^2}$

21. Divide $\frac{m^2-n^2}{b+c}$ by $am-an$. . . . . Ans. $\frac{m+n}{ab+ac}$.

22. Divide $\frac{a^3+b^3}{2+3x}$ by $ab+b^2$. . . . . Ans. $\frac{a^2-ab+b^2}{2b+3bx}$.

23. Divide $\frac{x^2-y^2}{3a}$ by x^2-xy. . . . . Ans. $\frac{x+y}{3ax}$.

24. Divide $\frac{a^3-b^3}{c}$ by a^2+ab+b^2. . . . . Ans. $\frac{a-b}{c}$.

ART. **142.**—To divide an integral or fractional quantity by a fraction.

1. How often is $\frac{2}{3}$ contained in 4, or what is the quotient of 4 divided by $\frac{2}{3}$?

4 is equal to $\frac{4}{1}=\frac{12}{3}$, and 2 thirds ($\frac{2}{3}$), is contained in 12 thirds ($\frac{12}{3}$), as often as 2 is contained in 12, that is, 6 times.

2. How often is $\frac{m}{n}$ contained in a?

a is equal to $\frac{a}{1}=\frac{na}{n}$, and $\frac{m}{n}$ is contained in $\frac{na}{n}$ as often as m is contained in na, that is $\frac{na}{m}$ times. Or, $\frac{1}{n}$ is contained in a, na times; hence, $m\times\frac{1}{n}$, or $\frac{m}{n}$ is contained $\frac{1}{m}$ as many times, that is, $\frac{na}{m}$ times.

3. How often is $\frac{2}{3}$ contained in $\frac{3}{4}$?

Here, $\frac{2}{3}=\frac{8}{12}$, and $\frac{3}{4}=\frac{9}{12}$, and 8 twelfths ($\frac{8}{12}$) is contained in 9 twelfths ($\frac{9}{12}$), as often as 8 is contained in 9, that is, $\frac{9}{8}=1\frac{1}{8}$ times.

4. How often is $\frac{m}{n}$ contained in $\frac{a}{c}$?

Reducing these fractions to a common denominator, $\frac{m}{n}=\frac{mc}{nc}$, and $\frac{a}{c}=\frac{na}{nc}$; now, $\frac{mc}{nc}$ is contained in $\frac{na}{nc}$ as often as mc is contained in na, that is, $\frac{na}{mc}$ times. This is the same result as that produced by multiplying $\frac{a}{c}$ by $\frac{m}{n}$ inverted, that is $\frac{a}{c}\times\frac{n}{m}=\frac{na}{mc}$.

An examination of each of these examples, will show that the process consists in *reducing the quantities to a common denominator, and then dividing the numerator of the dividend, by the numerator of the divisor.* But, as the common denominator of the fraction is not used in performing the division, the result will be the same as if we invert the divisor, and proceed as in multiplication. Hence, the

RULE,

FOR DIVIDING AN INTEGRAL OR FRACTIONAL QUANTITY BY A FRACTION.

Reduce both dividend and divisor to the form of a fraction; then invert the terms of the divisor, and multiply the numerators together for a new numerator, and the denominators together for a new denominator.

NOTE.—After inverting the divisor, the work may be abbreviated, by canceling all the factors common to both terms of the result.

EXAMPLES.

1. Divide 4 by $\frac{a}{3}$. . . . . Ans. $\frac{12}{a}$.

2. Divide 4 by $\frac{3}{a}$. . . . . Ans. $\frac{4a}{3}$.

3. Divide a by $\frac{1}{4}$. . . . . Ans. $4a$.

4. Divide ab^2 by $\frac{2ab}{5c}$. . . . . Ans. $\frac{5bc}{2}$.

5. Divide a^2-b^2 by $\frac{2(a+b)}{3a}$. . . . . Ans. $\frac{3a(a-b)}{2}$.

6. Divide $\frac{a}{3}$ by $\frac{c}{2}$. . . . . Ans. $\frac{2a}{3c}$.

REVIEW.—142. How do you divide an integral or fractional quantity by a fraction? Explain the reason of this rule, by analyzing an example. When, and how, can the work be abbreviated?

7. Divide $\frac{3a}{x}$ by $\frac{x}{3}$. . . . . Ans. $\frac{9a}{x^2}$.

8. Divide $\frac{a^2b}{cd}$ by $\frac{ab}{d}$. . . . . Ans. $\frac{a}{c}$.

9. Divide $\frac{x^2y}{3a}$ by $\frac{xy^2}{2b}$. . . . . Ans. $\frac{2bx}{3ay}$.

10 Divide $\frac{6ax}{5}$ by $\frac{4x}{3}$. . . . . Ans. $\frac{9a}{10}$.

11 Divide $\frac{3a^2x}{7}$ by $\frac{3ax^2}{14}$. . . . . Ans. $\frac{2a}{x}$.

12. Divide $\frac{16ax}{5}$ by $\frac{4x}{15}$. . . . . Ans. $12a$.

13 Divide $\frac{6z+4}{5}$ by $\frac{3z+2}{4y}$. . . . . Ans. $\frac{8y}{5}$

14. Divide $\frac{a^2-b^2}{5}$ by $\frac{a+b}{a}$. . . . . Ans. $\frac{a(a-b)}{5}$.

15. Divide $\frac{z^2-4}{6}$ by $\frac{z-2}{2}$. . . . . Ans. $\frac{z+2}{3}$.

16. Divide $\frac{x^2-2xy+y^2}{ab}$ by $\frac{x-y}{bc}$. . . . . Ans. $\frac{cx-cy}{a}$.

17. Divide $\frac{m^2-n^2}{3}$ by $\frac{m+n}{6}$. . . . . Ans. $2m-2n$.

18. Divide $\frac{a}{a^2-1}$ by $\frac{a+1}{a-1}$. . . . . Ans. $\frac{a}{a^2+2a+1}$.

19. Divide $\frac{4a+12}{9}$ by $\frac{3a+9}{18b}$. . . . . Ans. $\frac{8b}{3}$.

20. Divide $\frac{2z+3}{x+y}$ by $\frac{10z+15}{x^2-y^2}$. . . . . Ans. $\frac{x-y}{5}$.

21. Divide $\frac{a-b}{a+b}$ by $\frac{a^2-b^2}{a^2+2ab+b^2}$. . . . . Ans. 1.

22. Divide $\frac{3(a^2-x^2)}{x}$ by $\frac{2(a+x)}{a-x}$. . . . . Ans. $\frac{3(a^2-2ax+x^2)}{2x}$

23. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{a+x}$. . . . . Ans. $\frac{2x}{a^2-ax+x^2}$.

ART. **143.**—To reduce a complex fraction to a simple one.

This may be regarded as a case of division, in which the dividend and the divisor are either fractions or mixed quantities.

Thus, $\frac{2\frac{1}{3}}{3\frac{1}{2}}$, is the same as to divide $2\frac{1}{3}$ by $3\frac{1}{2}$.

Also $\frac{a+\frac{b}{c}}{m+\frac{n}{r}}$, is the same as to divide $a+\frac{b}{c}$ by $m+\frac{n}{r}$.

$$2\tfrac{1}{3}\div 3\tfrac{1}{2}=\frac{7}{3}\div\frac{7}{2}=\frac{7}{3}\times\frac{2}{7}=\frac{2}{3}.$$

$$\left(a+\frac{b}{c}\right)\div\left(m+\frac{n}{r}\right)=\frac{ac+b}{c}\div\frac{mr+n}{r}=\frac{ac+b}{c}\times\frac{r}{mr+n}=\frac{acr+br}{cmr+cn}$$

In the same manner, let the following examples be solved.

1. Reduce $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ to a simple fraction. . . . . Ans. $\dfrac{ad}{bc}$.

2. Reduce $\dfrac{3\frac{1}{2}}{\frac{a}{3}}$ to a simple fraction. . . . . Ans. $\dfrac{21}{2a}$.

3. Reduce $\dfrac{\frac{r}{m}}{n}$ to a simple fraction. . . . . Ans. $\dfrac{r}{mn}$.

4. Reduce $\dfrac{r}{\frac{m}{n}}$ to a simple fraction. . . . . Ans. $\dfrac{rn}{m}$.

5. Reduce $\dfrac{a+\frac{1}{c}}{m}$ to a simple fraction. . . . . . Ans. $\dfrac{ac+1}{cm}$.

6. Reduce $\dfrac{m}{a+\frac{1}{c}}$ to a simple fraction. . . . . . Ans. $\dfrac{cm}{ac+1}$.

A complex fraction may also be reduced to a simple one, by multiplying both terms by the least common multiple of the denominators of the fractional parts of each term. Thus, we may reduce $\dfrac{4\frac{1}{3}}{5\frac{1}{2}}$ to a simple fraction, by multiplying both terms by 6, the least common multiple of 2 and 3; the result is $\frac{26}{33}$. In some cases this is a shorter method, than by division. Either method may be used.

ART. **144.**—Resolution of fractions into series.

An *infinite* series consists of an unlimited number of terms, which observe the same law.

The *law* of a series is a relation existing between its terms, so that when some of them are known, the succeeding terms may be easily derived.

REVIEW.—143. How do you reduce a complex fraction to a simple one, by division? How, by multiplication?

Thus, in the infinite series, $1-ax+a^2x^2-a^3x^3+a^4x^4$, &c., any term may be found, by multiplying the preceding term by $-ax$.

Any proper algebraic fraction, whose denominator is a polynomial, can, by division, be resolved into an infinite series; for, the numerator is a dividend, and the denominator a divisor, so related to each other, that the process of division never can terminate, and the quotient will, therefore, be an infinite series. After a few of the terms of the quotient are found, the law of the series will, in general, be easily seen, so that the succeeding terms may be found without continuing the division.

EXAMPLES.

1. Convert the fraction $\frac{1}{1-x}$ into an infinite series.

$$\begin{array}{l|l} 1 & 1-x \\ \cline{2-2} \underline{1-x} & 1+x+x^2+x^3+, \text{ \&c.} \\ \quad +x & \\ \quad \underline{+x-x^2} & \\ \qquad +x^2 & \\ \qquad \underline{x^2-x^3} & \\ \qquad\quad \underline{+x^3} & \end{array}$$

The law of this series evidently is, that each term is equal to the preceding term, multiplied by $+x$.

From this, it appears, that the fraction $\frac{1}{1-x}$, is equal to the infinite series, $1+x+x^2+x^3+x^4+$, &c.

In a similar manner, let each of the following fractions be resolved into an infinite series, by division.

2. $\frac{1}{1+x}=1-x+x^2-x^3+x^4-$, &c., to infinity.

3. $\frac{ax}{a-x}=x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+$, &c.

4. $\frac{1+x}{1-x}=1+2x+2x^2+2x^3+$, &c.

5. $\frac{1-x}{1+x}=1-2x+2x^2-2x^3+$, &c.

6. $\frac{x+2}{x+1}=1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}$, &c.

REVIEW.—144. What is an infinite series? What is the law of a series? Give an example. Why can any proper algebraic fraction, whose denominator is a polynomial, be resolved into an infinite series, by division?

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE.

DEFINITIONS AND ELEMENTARY PRINCIPLES.

ART. **145.**—The most useful part of Algebra, is that which relates to the solution of problems. This is performed by means of equations.

An equation is an Algebraic expression, stating the equality between two quantities.

Thus, $x-3=4$, is an equation, stating, that if 3 be subtracted from x, the remainder will be equal to 4.

ART. **146.**—Every equation is composed of two parts, separated from each other by the sign of equality. The quantity on the left of the sign of equality, is called the *first member*, or side of the equation. The quantity on the right, is called the *second member*, or side. The members or quantities are each composed of one or more terms.

ART. **147.**—There are generally two classes of quantities in an equation, the *known* and the *unknown*. The known quantities are represented either by numbers, or the first letters of the alphabet, as a, b, c, &c.; and the unknown quantities by the last letters of the alphabet, as x, y, z, &c.

ART. **148.**—Equations are divided into degrees, called *first*, *second*, *third*, and so on. The degree of an equation, depends on the highest power of the unknown quantity which it contains.

An equation which contains no power of the unknown quantity higher than the first, is called *an equation of the first degree.*

Thus, $2x+5=9$, and $ax+b=c$, are equations of the first degree. Equations of the first degree are usually called Simple Equations.

An equation in which the highest power of the unknown quantity is of the second degree, that is, a square, is called *an equation of the second degree*, or *a quadratic equation.*

REVIEW.—145. What is an equation? Give an example. 146. Of how many parts is every equation composed? How are they separated? What is the quantity on the left of the sign of equality called? On the right? Of what is each member composed? 147. How many classes of quantities are there in an equation? How are the known quantities represented? How are the unknown quantities represented? 148. How are equations divided? On what does the degree of an equation depend? What is an equation of the first degree? Give an example. What are equations of the first degree usually called? What is an equation of the second degree? Give an exam ple. What are equations of the second degree usually called.

Thus, $4x^2-7=29$, and $ax^2+bx=c$, are equations of the second degree.

In a similar manner, we have equations of the *third* degree, *fourth* degree, &c.; the degree of the equation being always the same as the highest power of the unknown quantity which it contains.

When any equation contains more than one unknown quantity, its degree is equal to the greatest sum of the exponents of the unknown quantities in any of its terms.

Thus, $xy+ax+by=c$, is an equation of the second degree.

$x^2y+x^2+cx=a$, is an equation of the third degree.

Art. **149.**—An *identical equation*, is one in which the two members are identical; or, one in which one of the members is the result of the operations indicated in the other.

Thus, $2x-1=2x-1$, $5x+3x=8x$, and $(x+2)(x-2)=x^2-4$, are identical equations.

Equations are also distinguished as *numerical* and *literal*. A *numerical* equation is one in which all the known quantities are expressed by numbers.

Thus, $x^2+2x=3x+7$, is a numerical equation.

A literal equation is one in which the known quantities are represented by letters, or by letters and numbers.

Thus, $ax-b=cx+d$, and $ax^2+bx=2x-5$, are literal equations.

Art. **150.**—Every equation is to be regarded as the statement, in algebraic language, of a particular question.

Thus, $x-3=4$, may be regarded as the statement of the following question: To find a number, from which, if 3 be subtracted, the remainder will be equal to 4.

If we add 3 to each member, we shall have $x-3+3=4+3$, or $x=7$.

An equation is said to be *verified*, when the value of the unknown quantity being substituted for it, the two members are rendered equal to each other.

Thus, in the equation $x-3=4$, if 7, the value of x, be substituted instead of it, we have $7-3=4$, or, $4=4$.

To *solve* an equation, is *to find the value of the unknown quantity;* or, to find a number, which being substituted for the unknown quantity, will render the two members identical.

Review.—148. When an equation contains more than one unknown quantity, to what is its degree equal? Give an example. 149. What is an identical equation? Give examples. What is a numerical equation? Give an example. What is a literal equation? Give an example. 150. How is every equation to be regarded? Give an example. When is an equation said to be verified? What do you understand, by solving an equation?

ART. **151.**—The value of the unknown quantity in any equation, is called the *root* of that equation.

SIMPLE EQUATIONS, CONTAINING BUT ONE UNKNOWN QUANTITY.

ART. **152.**—The operations that we employ, to find the value of the unknown quantity in any equation, are founded on this evident principle: *If we perform exactly the same operation on two equal quantities, the results will be equal.* This principle, or axiom, may be otherwise stated, as follows:

1. *If, to two equal quantities, the same quantity be added, the sums will be equal.*
2. *If, from two equal quantities, the same quantity be subtracted, the remainders will be equal.*
3. *If two equal quantities be multiplied by the same quantity, the products will be equal.*
4. *If two equal quantities be divided by the same quantity, the quotients will be equal.*
5. *If two equal quantities be raised to the same power, the results will be equal.*
6. *If the same root of two equal quantities be extracted, the results will be equal.*

REMARK.—An axiom is a self-evident truth. The preceding axioms are the foundation of a large portion of the reasoning in mathematics.

ART. **153.**—There are two operations of frequent use in the solution of equations. These are, first, *to clear an equation of fractions;* and, second, *to transpose the terms, in order to find the value of the unknown quantity.* These are named in the order in which they are generally used, in the solution of an equation; we shall, however, first consider the subject of

TRANSPOSITION.

Suppose we have the equation $2x-3=x+5$.

Since, by the preceding principle, the equality will not be affected, by adding the same quantity to both members; or, by subtracting the same quantity from both members; if we add 3 to each member, we have $2x-3+3=x+5+3$.

If we subtract x from each member, we have

$$2x-x-3+3=x-x+5+3.$$

REVIEW.—151. What is the root of an equation? 152. Upon what principle are the operations founded, that are used in solving an equation? What are the axioms which this principle embraces? 153. What two operations are frequently used, in the solution of equations?

But, $-3+3$ cancel each other; so, also, do $x-x$; omitting these, we have $2x-x=5+3$.

Now, the result is the same as if we had removed the terms -3 and $+x$, to the opposite members of the equation, and, at the same time, *changed* their signs.

Again, take the equation $ax+b=c-dx$.

If we subtract b from each side, and add dx to each side, we have $ax+dx=c-b$.

But, this result is also the same as if we had removed the terms $+b$ and $-dx$ to the opposite members of the equation, and, at the same time, changed their signs. Hence,

Any quantity may be transposed from one side of an equation to the other, if, at the same time, its sign be changed.

TO CLEAR AN EQUATION OF FRACTIONS.

ART. **154.**—1. Let it be required to clear the following equation of fractions.

$$\frac{x}{2}+\frac{x}{3}=5.$$

Since the first term is divided by 2, if we multiply it by 2, the divisor will be removed; but if we multiply the first term by 2, we must multiply all the other terms by 2, in order to preserve the equality of the members. Multiplying both sides by 2, we have

$$x+\frac{2x}{3}=10.$$

Again, since the second term is divided by 3, if we multiply it by 3, the divisor will be removed; but, if we multiply the second term by 3, we must multiply all the terms by 3, in order to preserve the equality of the members. Multiplying both sides by 3, we have $3x+2x=30$.

Instead of multiplying first by 2, and then by 3, it is plain that we might have multiplied at once, by 2×3, that is, by the product of the denominators.

2. Again, let it be required to clear the following equation of fractions.

$$\frac{x}{ab}+\frac{x}{bc}=d.$$

Since the first term is divided by ab, if we multiply it by ab, the divisor will be removed; but, if we multiply the first term by ab, we must multiply all the other terms by ab, in order to preserve the equality of the members.

REVIEW.—153. How may a quantity be transposed from one member of an equation to the other? Explain the principle of transposition by an example.

Again, since the second term is divided by bc, if we multiply it by bc, the divisor will be removed; but, if we multiply the second term by bc, we must multiply all the other terms by bc, in order to preserve the equality of the members. Hence, if we multiply all the terms on both sides, by $ab\times bc$, the equation will be cleared of fractions.

Instead, however, of multiplying every term by $ab\times bc$, it is evident, that if each term be multiplied by such a quantity as will contain the denominators without a remainder, that all the denominators will be removed. This quantity is, evidently, the least common multiple of the denominators, which, in this case, is abc; then, multiplying both sides of the equation by abc, we have $cx+ax=abcd$. Hence, the

RULE,

FOR CLEARING AN EQUATION OF FRACTIONS.

Find the least common multiple of all the denominators, and multiply each term of the equation by it.

Clear the following equations of fractions.

1. $\frac{x}{2}+\frac{x}{3}=5$. . . . . Ans. $3x+2x=30$.
2. $\frac{x}{3}-\frac{x}{4}=2$. . . . . Ans. $4x-3x=24$.
3. $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=1$. . . . . Ans. $20x+15x+12x=60$.
4. $\frac{x}{4}+\frac{x}{8}-\frac{x}{6}=\frac{5}{12}$. . . . . Ans. $6x+3x-4x=10$.
5. $\frac{x}{3}-\frac{x}{5}+\frac{x}{10}=\frac{7}{10}$. . . . . Ans. $10x-6x+3x=21$
6. $\frac{x}{2}-4=\frac{x}{3}+6$. . . . . Ans. $3x-24=2x+36$.
7. $\frac{5x}{8}-\frac{5}{6}=\frac{3}{4}+\frac{7x}{12}$. . . . . . . Ans. $15x-20=18+14x$.
8. $\frac{x-4}{3}-\frac{2}{5}=\frac{6-x}{10}-4$. Ans. $10x-40-12=18-3x-120$.
9. $\frac{2x-3}{4}+\frac{x}{7}=\frac{x-3}{2}+\frac{5}{14}$. Ans. $14x-21+4x=14x-42+10$
10. $x-\frac{x-3}{2}=5-\frac{x+4}{3}$. . . . . Ans. $6x-3x+9=30-2x-8$
11. $\frac{x}{a}+\frac{x-5}{2}=b$. . . . . Ans. $2x+ax-5a=2ab$.
12. $\frac{4}{x-3}+\frac{2a}{3}=\frac{3}{4}$. . . . . Ans. $48+8ax-24a=9x-27$.

REVIEW.—154. How do you clear an equation of fractions? Explain the principle by an example.

13. $\frac{x+1}{x-3}+\frac{3-c}{a-b}=a.$

Ans. $ax-bx+a-b+3x-cx-9+3c=a^2x-abx-3a^2+3ab$

14. $\frac{x}{a+b}+\frac{x}{a-b}=\frac{c}{a^2-b^2}$. . . . . Ans. $ax-bx+ax+bx=c$

15. $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}=h$. . . . . . . Ans. $adf+bcf+bde=bdfhx$.

SOLUTION OF EQUATIONS OF THE FIRST DEGREE, CONTAINING ONLY ONE UNKNOWN QUANTITY.

Art. **155.**—The unknown quantity in an equation may be combined with the known quantities, either by Addition, Subtraction, Multiplication, or Division; or, by two or more of these different methods.

1. Let it be required to find the value of x, in the equation

$$x+3=5,$$

where the unknown quantity is connected by addition.

By *subtracting* 3 from each side, we have $x=5-3=2$.

2. Let it be required to find the value of x, in the equation

$$x-3=5,$$

where the unknown quantity is connected by subtraction.

By *adding* 3 to each side, we have $x=5+3=8$.

3. Let it be required to find the value of x, in the equation

$$3x=15,$$

where the unknown quantity is connected by multiplication.

By *dividing* each side by 3, we have $x=\frac{15}{3}=5$.

4. Let it be required to find the value of x, in the equation

$$\frac{x}{3}=2,$$

where the unknown quantity is connected by division.

By *multiplying* each side by 3, we have $x=2\times3=6$.

From the solution of these examples, we see, that *when the unknown quantity is connected by addition, it is to be separated by subtraction. When it is connected by subtraction, it is to be separated by addition. When it is connected by multiplication, it is to be separated by division. And, when it is connected by division, it is to be separated by multiplication.*

5. Find the value of x, in the equation

$$3x-3=x+5.$$

By transposing the terms -3 and x, we have

$$3x-x=5+3,$$

reducing, $2x=8$,

dividing by 2, $x=\frac{8}{2}=4$.

Let this value of x be substituted instead of x, in the original equation, and, if it is the true value, it will render the two members equal to each other.

Original equation, ... $3x-3=x+5$.

Substituting 4 in the place of x, it becomes

$$3\times4-3=4+5, \text{ or } 9=9.$$

The operation of substituting the *value* of the unknown quantity instead of itself, in the original equation, to see if it will render the two members equal to each other, is called *verification.*

The preceding equation may be solved thus:

$3x-3=x+5$. By adding 3 to each member, we have $3x-3+3=x+5+3$. By subtracting x from each member, we have $3x-x-3+3=x-x+5+3$.

But $-3+3$ cancel each other; so, also, do x and $-x$; omitting these, and then reducing, we have $2x=8$.

Dividing each member by 2, $x=\frac{8}{2}=4$.

REMARK.—The pupil will perceive that the two methods of solution are the same in principle. In the first, we use transposition, to remove the known quantity from the left member to the right, and the unknown quantity from the right member to the left. In the second, the same thing is done, by adding equals to each member, and subtracting equals from each member—this being the principle on which transposition is founded. It is recommended to the teacher, to use the latter method until the principle is well understood by the pupil, after which the first method may be used exclusively.

6. Find the value of x in the equation $x-\frac{x-2}{3}=4+\frac{x+2}{5}$.

Multiplying both sides by 15, the least common multiple of the denominators, we have $15x-(5x-10)=60+3x+6$.

or, $15x-5x+10=60+3x+6$.

by transposition, $15x-5x-3x=60+6-10$.

reducing, $7x=56$.

dividing, $x=8$.

7. Find the value of x in the equation $\frac{x}{b}-d=\frac{x}{a}+c$.

multiplying both sides by ab, $ax-abd=bx+abc$.

transposing, $ax-bx=abc+abd$.

separating into factors, $(a-b)x=ab(c+d)$.

dividing by $(a-b)$, $x=\frac{ab(c+d)}{a-b}$.

REVIEW.—155. What are the methods by which the unknown quantity in an equation may be combined with known quantities? Give examples. When the unknown quantity is connected by addition, how can it be separated? When, by subtraction? By multiplication? By division? What is verification?

From the preceding examples and illustrations, we derive the

RULE,

FOR THE SOLUTION OF AN EQUATION OF THE FIRST DEGREE.

1. *If necessary, clear the equation of fractions; perform all the operations indicated; and transpose all the terms containing the unknown quantity to one side, and the known quantities to the other.*

2. *Reduce each member to its simplest form, and divide both sides by the coëfficient of the unknown quantity.*

EXAMPLES FOR PRACTICE.

NOTE.—Let the pupil verify the value of the unknown quantity in each example.

1. $3x-5=2x+7$. . . . . Ans. $x=12$.
2. $3x-8=16-5x$. . . . . Ans. $x=3$.
3. $5x-7=3x+15$ . . . . Ans. $x=11$.
4. $3x-25=-x-9$. . . . . Ans. $x=4$.
5. $15-2x=6x-25$. . . . . Ans. $x=5$.
6. $5(x+1)+6(x+2)=9(x+3)$. . . . . Ans. $x=5$.
7. $4(5x-3)-64(3-x)-3(12x-4)=96$. . . . Ans. $x=6$.
8. $10(x+5)+8(x+4)=5(x+13)+121$. . . . . Ans. $x=8$.

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9. $\frac{x}{2}-2=5-\frac{x}{5}$. . . . . Ans. $x=10$.
10. $\frac{x}{2}-\frac{x}{4}+7=\frac{x}{8}-\frac{x}{6}+10\frac{1}{2}$. . . . . Ans. $x=12$.
11. $x+\frac{x}{2}+\frac{3x}{4}=18$. . . . . Ans. $x=8$.
12. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=14$. . . . . Ans. $x=24$.
13. $\frac{3x}{4}+\frac{2x}{3}-\frac{x}{6}=2\frac{1}{2}$. . . . . Ans. $x=2$.
14. $\frac{x-2}{4}-2=1-\frac{x+7}{3}$. . . . . Ans. $x=2$.
15. $\frac{3x+1}{2}-\frac{2x}{3}=10+\frac{x-1}{6}$. . . . . Ans. $x=14$.
16. $\frac{x+2}{3}-\frac{x-3}{4}=x-2-\frac{x-1}{2}$. . . . . Ans. $x=7$.
17. $\frac{3x-2}{4}-\frac{4-x}{2}=2x-\frac{7x-2}{3}$. . . . . Ans. $x=2$.

REVIEW.—155. What is the rule for the solution of an equation of the first degree, containing one unknown quantity?

18. $\frac{4}{5}x-\frac{5}{4}x+18=\frac{1}{9}(4x+1)$. . . . . Ans. $x=20$

19. $\frac{5x}{x+4}=1$. . . . . Ans. $x=1$.

20. $2x-\frac{x-2}{10}=x+\frac{x+18}{15}$. . . . . Ans. $x=1\frac{1}{5}$.

21. $\frac{3}{4}-\frac{x-2}{3}=\frac{5}{4}-\frac{x+3}{4}$. . . . . Ans. $x=11$.

22. $\frac{x+3}{4}-\frac{x-3}{5}=\frac{x-5}{2}-2$. . . . . Ans. $x=13$.

23. $\frac{x-3}{4}+6-\frac{x-1}{5}=\frac{x-5}{2}+3$. . . . . Ans. $x=11$.

24. $2x-\frac{2x+11}{5}-\frac{4x-6}{11}=-\frac{7-8x}{7}$. . . . . Ans. $x=7$.

25. $\frac{x+7}{3}-5\frac{3}{4}=\frac{2x+5}{7}+\frac{10-5x}{8}$. . . . . Ans. $x=8$.

26. $\frac{x}{8}+\frac{2(x-1)}{5}=\frac{7x-4}{15}-\frac{x-1}{60}$. . . . . Ans. $x=2$.

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27. $4x-b=2x-d$. . . . . Ans. $x=\frac{b-d}{2}$.

28. $ax+b=cx+d$. . . . . Ans. $x=\frac{d-b}{a-c}$.

29. $ax-bx=d-cx$. . . . . Ans. $x=\frac{d}{a+c-b}$.

30. $ax-bx=c+dx-e$. . . . . Ans. $x=\frac{c-e}{a-b-d}$.

31. $7+9a-5x=6x+5ax$. . . . . Ans. $x=\frac{9a+7}{5a+11}$.

32. $b(a-bx)+c(ax-c)=bc$. . . . . Ans. $x=\frac{ab-bc-c^2}{b^2-ac}$.

33. $(a+b)(b-x)+(a-b)(a+x)=c^2$. . . . Ans. $x=\frac{a^2+b^2-c^2}{2b}$.

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34. $\frac{x}{a}+\frac{x}{b}=c$. . . . . Ans. $x=\frac{abc}{a+b}$.

35. $\frac{ab}{x}=bc+\frac{1}{x}$. . . . . Ans. $x=\frac{ab-1}{bc}$.

36. $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}=1$. . . . . Ans. $x=a+b+c$.

37. $\frac{x}{a}+c=\frac{x}{b}-d$. . . . . Ans. $x=\frac{ab(c+d)}{a-b}$.

38. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d.$ . . . . Ans. $x=\frac{abcd}{ab+ac+bc}$.

39. $\frac{x}{a-b}=\frac{x}{a+b}+1.$ . . . . Ans. $x=\frac{a^2-b^2}{2b}$.

40. $\frac{x}{a}+\frac{x}{b}-\frac{x}{c}=1.$ . . . . Ans. $x=\frac{abc}{ac+bc-ab}$.

41. $\frac{ab}{x}+\frac{ac}{x}+\frac{bc}{x}=2.$ . . . . Ans. $x=\frac{1}{2}(ab+ac+bc)$.

42. $\frac{a}{x}+\frac{b}{c}-\frac{d}{e}=0.$ . . . . Ans. $x=\frac{ace}{cd-be}$

43. $\frac{x-a}{b}-\frac{x-b}{a}=\frac{b}{a}.$ . . . . Ans. $x=\frac{a^2}{a-b}$

44. $\frac{1+x}{1-x}=1+\frac{1}{a}.$ . . . . Ans. $x=\frac{1}{2a+1}$

45. $\frac{a^2}{x}=ab+b+\frac{1}{x}.$ . . . . Ans. $x=\frac{a-1}{b}$

46. $\frac{a-b}{x-c}=\frac{a+b}{x+2c}.$ . . . . Ans. $x=\frac{3ac-bc}{2b}$.

QUESTIONS PRODUCING SIMPLE EQUATIONS, CONTAINING ONLY ONE UNKNOWN QUANTITY.

ART. **156.**—The solution of a problem, by Algebra, consists of two distinct parts.

1st. *To express the conditions of the problem in Algebraic language; that is, to form the equation.*

2d. *To solve the equation; that is, to find the value of the unknown quantity.*

With pupils, the most difficult part of the operation of solving a question, is to form the equation, by the solution of which the value of the unknown quantity is to be found. Sometimes, the statement of the question furnishes the equation directly; and, sometimes, it is necessary, from the conditions given, to deduce others, from which to form the equation. When the conditions furnish the equation directly, they are called *explicit* conditions. When the conditions are deduced from those given in the question, they are called *implied* conditions.

It is impossible to give a precise rule, by means of which every question may be readily stated in the form of an equation. The first point, is, to understand fully the nature of the question, so as to be able to prove whether any proposed answer is correct.

REVIEW.—156. Of what two parts does the solution of a problem by Algebra, consist? What are explicit conditions? What are implied conditions?

After this, the equation, by the solution of which the value of the unknown quantity is to be found, may generally be formed by the following

RULE.

Denote the required quantity, by one of the final letters of the alphabet; then, by means of signs, indicate the same operations that it would be necessary to make on the answer, to verify it.

REMARKS.—1st. In solving a question, it is necessary to understand the principles of the science which it involves, at least so far as they relate to the question under consideration. Thus, when a problem embraces the consideration of Ratio or Proportion, in order to solve it, the pupil must be familiar with these subjects. In the following examples, the learner is supposed to be acquainted with Ratio and Proportion, as far as they are taught in Arithmetic. (See Ratio and Proportion, Ray's Arithmetic, Part III.)

2d. The operations concerned in the solution of an equation, involve the removal of coëfficients, the removal of denominators, and the transposition of quantities. The first six of the following examples, and also those from the 16th to the 44th inclusive, are arranged with reference to these operations.

EXAMPLES.

1. There are two numbers, the second of which is three times the first, and their sum is 48; what are the numbers?

Let $x=$ the first number.

Then, by the first condition, $3x=$ the second.

And, by the second condition, $x+3x=48$.

Reducing, $4x=48$.

Dividing by 4, $x=12$, the smaller number.

Then, $3x=36$, the larger number.

Proof, or *verification*. $12+36=48$.

2. A father said to his son, "The difference of our ages is 48 years, and I am 5 times as old as you." What were their ages?

Let $x=$ the son's age.

Then $5x=$ the father's age.

And $5x-x=48$.

Reducing, $4x=48$.

Dividing, $x=12$, the son's age.

Then $5x=60$, the father's age.

Verification. $60-12=48$, the difference of their ages.

3. What number is that, to which, if its third part be added, the sum will be 16?

Let $x=$ the required number.

REVIEW.—156. By what general rule, may the equation of a problem be found?

Then the third part of it will be represented by $\frac{x}{3}$.

And, by the conditions of the question, we have the equation

$$x+\frac{x}{3}=16.$$

Multiplying it by 3, to clear it of fractions, $3x+x=48$.
Reducing, $4x=48$.
Dividing, $x=12$.

Verification. $12+\frac{12}{3}=12+4=16$; which shows that the value found is correct, since it satisfies the conditions of the question.

NOTE.—The pupil should *verify* the answer in every example.

4. What number is that, which being increased by its half, and then diminished by its two thirds, the remainder will be equal to 105.

Let $x=$ the number.

Then the one half will be represented by $\frac{x}{2}$, and the two thirds by $\frac{2x}{3}$.

And, by the question $x+\frac{x}{2}-\frac{2x}{3}=105$.

Multiplying by 6, $6x+3x-4x=630$.
Reducing, $5x=630$.
Dividing, $x=126$. Ans.

When the numbers contained in a solution are large, it is sometimes better to indicate the multiplication, than to perform it.

The preceding solution may be given thus:

$$x+\frac{x}{2}-\frac{2x}{3}=105$$
$$6x+3x-4x=105\times6$$
$$5x=105\times6$$
$$x=\ 21\times6=126.$$

5. It is required to divide a line 25 inches long, into two parts, so that the greater shall be 3 inches longer than the less.

Let $x=$ the length of the smaller part.
Then $x+3=$ the greater part.
And by the question, $x+x+3=25$.
Reducing, $2x+3=25$.
Transposing 3, $2x=25-3=22$.
Dividing, $x=11$, the smaller part.
And $x+3=14$, the greater part.

6. It is required to divide 68 dollars between A, B, and C, so that B shall have 5 dollars more than A, and C 7 dollars more than B.

Let $x=$ A's share.

Then $x+5=$ B's share.

And $x+12=$ C's share. Then, by the terms of the question, we have $x+(x+5)+(x+12)=68$.

Reducing, $3x+17=68$.

Transposing, $3x=68-17=51$.

Dividing, $x=17$, A's share.

$x+5\ =22$, B's share.

$x+12=29$, C's share.

7. What number is that, which being added to its third part, the sum will be equal to its half added to 10.

Let x represent the number.

Then, the number, with its third part, is represented by $x+\frac{x}{3}$; and its half, added to 10, is expressed by $\frac{x}{2}+10$. By the conditions of the question, these are equal; that is, $x+\frac{x}{3}=\frac{x}{2}+10$.

Multiplying by 6, $6x+2x=3x+60$.

Reducing and transposing, $8x-3x=60$.

$5x=60$.

Dividing, $x=12$.

Verification. $12+\frac{12}{4}=\frac{12}{3}+10$. Or $16=16$, according to the conditions.

Hereafter, we shall, in general, omit the terms, *transposing*, *dividing*, &c., as the various steps of the solution will be evident by inspection.

8. A cistern was found to be one third full of water, and after emptying into it 17 barrels more, it was found to be half full; what number of barrels will it contain when full?

Let $x=$ the number of barrels the cistern will contain

Then $\frac{x}{3}+17=\frac{x}{2}$.

$2x+102=3x$

$102=x$

Or, by first transposing $3x$ and 102, we have $-x=-102$; and multiplying both sides by -1, we have $x=102$.

The unknown quantity, when its value is found, is generally made to stand on the left side of the sign of equality; it is not material, however, which side it occupies, since, by transposition, it can be readily removed to the other. In effecting the transposition of $102=x$, so as to bring the x on the left side, we have made it to consist of two steps; it is, however, generally made in one; the transposition, and multiplying by -1, being both made in one line at the same time.

NOTE.—Multiplying by —1 is the same as changing all the signs of both members of the equation.

9. A cistern is supplied with water, by two pipes; the less alone can fill it in 40 minutes, and the greater in 30 minutes; in what time will they fill it, both running at once?

Let $x=$ the number of min. in which both together can fill it.

Then $\frac{1}{x}=$ the part which both can fill in 1 minute.

Since the less can fill it in 40 minutes, it fills $\frac{1}{40}$ of it in 1 minute. Since the greater can fill it in 30 minutes, it fills $\frac{1}{30}$ of it in 1 minute. Hence, the part of the cistern which both can fill in 1 minute, is represented by $\frac{1}{40}+\frac{1}{30}$, and also, by $\frac{1}{x}$.

Hence, $\frac{1}{40}+\frac{1}{30}=\frac{1}{x}$.

Multiply both sides by $120x$, and we have $3x+4x=120$.

$$7x=120.$$

$$x=\tfrac{120}{7}=17\tfrac{1}{7}\text{ min.}$$

10. A laborer, A, can perform a piece of work in 5 days, B can do the same in 6 days, and C in 8 days; in what time can the three together perform the same work?

Let $x=$ the number of days in which all three can do it.

Then $\frac{1}{x}=$ the part which all can do in 1 day.

If A can do it in 5 days, he does $\frac{1}{5}$ of it in 1 day.
If B " " 6 " " $\frac{1}{6}$ " "
If C " " 8 " " $\frac{1}{8}$ " "

Hence, the part of the work done by A, B, and C in 1 day, is represented by $\frac{1}{5}+\frac{1}{6}+\frac{1}{8}$, and also, by $\frac{1}{x}$.

Hence, $\frac{1}{5}+\frac{1}{6}+\frac{1}{8}=\frac{1}{x}$.

Or, $24x+20x+15x=120$.

$$59x=120$$

$$x=\tfrac{120}{59}=2\tfrac{2}{59}\text{ days.}$$

11. How many pounds of sugar at 5 cents, and at 9 cents per pound, must be mixed, to make a box of 100 pounds, at 6 cents per pound.

Let $x=$ the number of pounds at 5 cents.
Then $100-x=$ the number of pounds at 9 cents.
Also, $5x=$ the value of the former.
And $9(100-x)=$ the value of the latter.
And $600=$ the value of the mixture.

But the value of the two kinds must be equal to the value of the mixture.

Therefore, $5x+9(100-x)=600$

$5x+900-9x=600$

$-4x=-300$

$x=75$, the number of pounds at 5 cents.

$100-x=25$, " " " " 9 cents.

12. A laborer was engaged for 30 days. For each day that he worked, he received 25 cents and his boarding; and, for each day that he was idle, he paid 20 cents for his boarding. At the expiration of the time, he received 3 dollars; how many days did he work, and how many days was he idle?

Let $x=$ the number of days he worked.

Then $30-x=$ the number of days he was idle.

Also $25x=$ wages due for work.

And $20(30-x)=$ the amount to be deducted for boarding.

Therefore, $25x-20(30-x)=300$

$25x-600+20x=300$

$45x=900$

$x=20=$ the number of days he worked.

$30-x=10=$ the number of days he was idle.

Proof. $25\times20=500$ cents, = wages.

$20\times10=\underline{200}$ cents, = boarding.

300 cents, = the remainder.

In solving this example, we reduce the 3 dollars to cents, in order that the quantities on both sides of the equation may be of the same denomination. For, as we can only add or subtract numbers of the same denomination, it is evident, that we can only compare quantities of the same name. Hence, *all the quantities, in both members of an equation, must be of the same denomination.*

13. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are equal to 3 of the hare's; how many leaps must the greyhound take, to catch the hare?

Let x be the number of leaps taken by the hound. Then, since the hare takes 4 leaps while the hound takes 3, the number of leaps taken by the hare, after the starting of the hound, will be $\frac{4x}{3}$; and the whole number of leaps taken by the hare, will be $\frac{4x}{3}+50$, which is equal, in extent, to the x leaps run by the hound. Now, if the length of the leaps taken by each were equal, we might put x equal to $\frac{4x}{3}+50$; but, by the question, 2 leaps of the

hound are equal to 3 of the hare's, that is, 1 leap of the hound is equal to $\frac{3}{2}$ leaps of the hare; hence, x leaps of the hound are equal to $\frac{3x}{2}$ leaps of the hare; and we have the equation

$$\frac{3x}{2}=\frac{4x}{3}+50.$$
$$9x=8x+300$$
$$x=300, \text{ leaps taken by the greyhound.}$$

14. The hour and minute hands of a watch are exactly together between 8 and 9 o'clock; required the time.

Let the number of minutes more than 40, be denoted by x; that is, let $x=$ the minutes from VIII to the point of coincidence, P; then, the hour hand moves from VIII to the point P, while the minute hand moves from XII to the same point; or, the former moves over x minutes, while the latter moves over $40+x$ minutes; but the minute hand moves 12 times as fast as the hour hand. Therefore, $12x=40+x$

$$11x=40$$
$$x=\tfrac{40}{11} \text{ minutes } =3 \text{ minutes, } 38\tfrac{2}{11} \text{ seconds.}$$

Hence, the required time is 43 minutes, $38\frac{2}{11}$ seconds after 8 o'clock.

15. A person spent one fourth of his money, and then received 5 dollars. He next spent one half of what he then had, and found that he had only 7 dollars remaining; what sum had he at first?

Let $x=$ the number of dollars he had at first. Then, after spending one fourth of that, and receiving 5 dollars, he had $x-\frac{x}{4}+5$, which being reduced, is equal to $\frac{3x}{4}+5$.

He now spent the half of this sum, or $\frac{1}{2}\left(\frac{3x}{4}+5\right)=\frac{3x}{8}+\frac{5}{2}$.

Therefore, $\frac{3x}{4}+5-\left(\frac{3x}{8}+\frac{5}{2}\right)=7;$

$$\text{or, } \frac{3x}{4}+5-\frac{3x}{8}-\frac{5}{2}=7;$$
$$\text{or, } \frac{3x}{4}-\frac{3x}{8}=2+\frac{5}{2};$$
$$\text{or, } 6x-3x=16+20;$$
$$3x=36$$
$$x=12. \quad \text{Ans.}$$

16. Divide 42 cents between A and B, giving to B twice as many as to A. Ans. A 14, B 28.

17. Divide the number 48 into three parts, so that the second may be twice, and the third three times the first.

Ans. 8, 16, and 24.

18. Divide the number 60 into 3 parts, so that the second may be three times the first, and the third double the second.

Ans. 6, 18, and 36.

19. A boy bought an equal number of apples, lemons, and oranges, for 56 cents; for the apples he gave 1 cent a piece, for the lemons 2 cents a piece, and for the oranges 5 cents a piece; how many of each did he purchase? Ans. 7.

20. A boy bought 5 apples and 3 lemons, for 22 cents; he gave as much for 1 lemon as for 2 apples; what did he give for each?

Ans. 2 cents for an apple, and 4 cents for a lemon.

21. The age of A is double that of B, the age of B is twice that of C, and the sum of all their ages is 98 years; what is the age of each? Ans. A 56 years, B 28 years, and C 14 years.

22. Four boys, A, B, C, and D, have, between them, 44 cents; of which A has a certain number, B has three times as many as A, C as many as A and one third as many as B, and D as many as B and C together; how many has each?

Ans. A 4, B 12, C 8, and D 20.

23. A man has 4 children, the sum of whose ages is 48 years, and the common difference of their ages is equal to twice that of the youngest; required their ages. Ans. 3, 9, 15, and 21 years.

24. Divide the number 55 into two parts, in proportion to each other as 2 to 3.

Let $2x=$ one part; then $3x=$ the other, since $2x$ is to $3x$ as 2 is to 3.

$$2x+3x=55$$
$$5x=55$$
$$x=11$$
$$\left.\begin{array}{r}2x=22\\3x=33\end{array}\right\}\text{Ans.}$$

Or thus: Let $x=$ one part; then $55-x=$ the other.

By the question, $x:55-x::2:3$. Then, since, in every proportion, the product of the means is equal to the product of the extremes, we have $3x=2(55-x)=110-2x$

$$5x=110$$
$$x=22, \text{ and } 55-x=33, \text{ as before.}$$

Or thus: Let $x=$ one part, then $\frac{3x}{2}=$ the other.

And $x+\frac{3x}{2}=55$.

$2x+3x=110$, from which $x=22$, and $\frac{3x}{2}=33$.

The first method avoids fractions, and is of such frequent application, that we may give this general direction:

When two or more unknown quantities in any problem, are to each other in a given ratio, it is best to assume each of them a multiple of some other unknown quantity, so that they shall have to each other the given ratio.

25. The sum of two numbers is 60, and the less is to the greater as 5 to 7; what are the numbers? Ans. 25 and 35.

26. Divide the number 60 into 3 parts, which shall be in proportion to each other as 2, 3, and 5. Ans. 12, 18, and 30.

27. Divide the number 92 into 4 parts, in proportion to each other as the numbers 3, 5, 7, and 8. Ans. 12, 20, 28, and 32.

28. Divide the number 60 into 3 such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{5}$ of the third, shall all be equal to each other. Ans. 12, 18, and 30.

This question is most conveniently solved, by putting $2x$, $3x$, and $5x$ for the parts, since the $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ of these are respectively equal to each other.

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29. What number is that whose half, third, and fourth part are together equal to 65? Ans. 60.

30. What number is that, $\frac{1}{5}$ of which is greater than $\frac{1}{7}$ by 4? Ans. 70.

31. The age of B is two and four fifth times the age of A, and the sum of their ages is 76 years; what is the age of each? Ans. A 20, B 56 years.

32. Divide 88 dollars between A, B, and C, giving to B $\frac{2}{3}$, and to C $\frac{3}{7}$ as much as to A. Ans. A \$42, B \$28, and C \$18.

33. Divide 440 dollars between three persons, A, B, and C, so that the share of A may be $\frac{3}{5}$ that of B, and the share of B $\frac{3}{4}$ that of C. Ans. A's share \$90, B's \$150, and C's \$200.

34. Four towns are situated in the order of the letters A, B, C, D. The distance from A to D is 120 miles; the distance from A to B is to the distance from B to C as 3 to 5; and one third of the distance from A to B, added to the distance from B to C, is three times the distance from C to D; how far are the towns apart?

Ans. A to B, 36 miles; B to C, 60 miles; C to D, 24 miles.

35. A merchant having engaged in trade with a certain capital, lost $\frac{1}{3}$ of it the 1st year; the 2d year he gained a sum equal to $\frac{2}{5}$ of what remained at the close of the 1st year; the 3d year he lost $\frac{1}{7}$ of what he had at the close of the 2d year, when he was worth \$1236. What was his original capital? Ans. \$1545.

36. The rent of a house this year, is greater, by 5 per cent., than it was last year; this year the rent is 168 dollars; what was it last year? Ans. \$160.

37. Divide the number 32 into 2 parts, so that the greater shall exceed the less by 6. Ans. 13 and 19.

38. At an election, the number of votes given for two candidates, was 256; the successful candidate had a majority of 50 votes; how many votes had each? Ans. 153 and 103.

39. Divide 1520 dollars between three persons, A, B, C, so that B may receive 100 dollars more than A, and C 270 dollars more than B; what is the share of each?

Ans. A $350, B $450, and C $720.

40. A company of 90 persons consists of men, women, and children; the men are 4 more than the women, and the children are 10 more than both men and women; what is the number of each? Ans. 18 women, 22 men, and 50 children.

41. After cutting off a certain quantity of cloth from a piece containing 45 yards, it was found that there remained 9 yards less than had been cut off; how many yards had been cut off?

Ans. 27.

42. What number is that, which, being multiplied by 7, gives a product as much greater than 20, as the number itself is less than 20? Ans. 5.

43. A person dying, left an estate of 6500 dollars, to be divided between his widow, 2 sons, and 3 daughters, so that each son shall receive twice as much as a daughter, and the widow 500 dollars less than all her children together; required the share of the widow, and of each son and daughter.

Ans. Widow $3000, each son $1000, and each daughter $500.

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44. Two men set out at the same time, one from London, and the other from Edinburgh; one goes 20, and the other 30 miles a day; in how many days will they meet, the distance being 400 miles? Ans. 8 days.

45. Two persons, A and B, depart from the same place, to go in the same direction; B travels at the rate of 3, and A at the rate of 5 miles an hour, but B has the start of A 10 hours; in how many hours will A overtake B? Ans. 15.

46. What number is that, of which one half and one third of it diminished by 44, is equal to one fifth of it diminished by 6?

Ans. 60.

47. A person being asked the time of day, replied, "If, to the time past noon, there be added its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{5}$, the sum will be equal to $\frac{1}{6}$ of the time to midnight; required the hour.

Ans. 50 min. P. M.

48. Divide the number 120 into two such parts, that the smaller may be contained in the greater $1\frac{1}{2}$ times. Ans. 48 and 72.

49. "I have a certain number in my mind," said A to B; "if I multiply it by 7, add 3 to the product, divide this by 2, and subtract 4 from the quotient, the remainder is 15." What is the number? Ans. 5.

50. What number is that, which, if you multiply it by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, will give the number itself? Ans. 54.

51. Two persons, A and B, engaged in trade, the capital of B being $\frac{2}{3}$ that of A; B gained, and A lost, 100 dollars; after which, if $\frac{5}{7}$ of what A had left, be subtracted from what B now has, the remainder will be 134 dollars; with what capital did each commence? Ans. A $786, B $524.

52. A man having spent 3 dollars more than $\frac{2}{3}$ of his money, had 7 dollars more than $\frac{1}{6}$ of it left; how many dollars had he at first? Ans. $75.

53. Two men, A and B, have the same annual income; A saves $\frac{1}{5}$ of his, but B spends 25 dollars per annum more than A, and at the end of 5 years finds he has saved 200 dollars; what is the annual income of each? Ans. $325.

54. In the composition of a quantity of gunpowder, $\frac{2}{3}$ of the whole, plus 10 pounds, was nitre; $\frac{2}{23}$ of the whole, plus 1 pound, was sulphur; and $\frac{1}{3}$ of the whole, minus 17 pounds, was charcoal; how many pounds of gunpowder were there? Ans. 69lb.

55. A person bought a chaise, horse, and harness, for 245 dollars; the horse cost 3 times as much as the harness, and the chaise cost 19 dollars less than $2\frac{2}{3}$ times as much as both horse and harness; what was the cost of each?

Ans. Harness $18, horse $54, chaise $173.

56. What two numbers are as 3 to 4, to each of which, if 4 be added, the sums will be to each other as 5 to 6? Ans. 6 and 8.

57. What two numbers are as 2 to 5, from each of which, if 2 be subtracted, the remainders will be to each other as 3 to 8?

Ans. 20 and 50.

58. The ages of two brothers are now 25 and 30 years, so that their ages are as 5 to 6; in how many years will their ages be as 8 to 9? Ans. 15.

How many years since their ages were as 1 to 2? A. 20 yrs.

59. A cistern has 3 pipes to fill it; by the first, it can be filled in $1\frac{1}{3}$ hours, by the second, in $3\frac{1}{3}$ hours, and by the third, in 5 hours; in what time can it be filled, by all three running at once? Ans. 48 min.

60. Find the time in which A, B, and C together, can perform a piece of work, which requires 7, 6, and 9 days respectively, when done singly. Ans. $2\frac{20}{53}$ days.

61. From a certain sum I took one third part, and put in its stead 50 dollars; next, from this sum I took the tenth part, and put in its stead 37 dollars; I then counted the money, and found I had 100 dollars; what was the original sum? Ans. \$30.

62. A teacher spent $\frac{2}{5}$ of his yearly salary for board and lodging, $\frac{1}{3}$ of the remainder for clothes, and $\frac{1}{5}$ of what remained, for books, and still saved 120 dollars per annum; what was his salary?

Ans. \$375.

63. A laborer was engaged for a year, at 80 dollars and a suit of clothes; after he had served 7 months, he left, and received for his wages, the clothes and 35 dollars; what was the value of the suit of clothes? Ans. \$28.

64. A man and his wife can drink a cask of wine in 6 days, and the man alone can drink it in 10 days; how many days will it last the woman? Ans. 15.

65. A steamboat, that can run 15 miles per hour with the current, and 10 miles per hour against it, requires 25 hours to go from Cincinnati to Louisville, and return; what is the distance between those cities? Ans. 150 miles.

66. A and B engaged in a speculation; A with 240 dollars, and B with 96 dollars; A lost twice as much as B, and, upon settling their accounts, it appeared, that A had 3 times as much remaining as B; what did each lose? Ans. A \$96, and B \$48.

67. In a mixture of wine and water, $\frac{1}{2}$ the whole, plus 25 gallons, was wine, and $\frac{1}{3}$ of the whole, minus 5 gallons, was water; required the quantity of each in the mixture.

Ans. 85 galls. of wine, and 35 galls. of water.

68. It is required to divide the number 91 into 2 such parts, that the greater, being divided by their difference, the quotient will be 7. Ans. 49 and 42.

69. It is required to divide the number 72 into 4 such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, the difference, the product, and the quotient shall all be equal.

Ans. 14, 18, 8, and 32.

Let the four parts be represented by $x-2$, $x+2$, $\frac{1}{2}x$, and $2x$.

70. A merchant having cut 19 yards from each of 3 equal pieces of silk, and 17 from another of the same length, found, that the remnants taken together, measured 142 yards; what was the length of each piece? Ans. 54 yds.

71. Suppose, that for every 10 sheep a farmer keeps, he should plow an acre of land, and allow 1 acre of pasture for every 4 sheep; how many sheep can the person keep, who farms 161 acres? Ans. 460.

72. It is required to divide the number 34 into 2 such parts, that if 18 be subtracted from the greater, and the less be subtracted from 18, the first remainder shall be to the second as 2 to 3. Ans. 22 and 12.

73. A person was desirous of giving 3 cents a piece to some beggars, but found that he had not money enough in his pocket by 8 cents; he therefore gave each of them 2 cents, and then had 3 cents remaining; required the number of beggars. Ans. 11.

74. A father distributed a number of apples among his children, as follows: to the first he gave $\frac{1}{2}$ the whole number, less 8; to the second $\frac{1}{2}$ the remainder, diminished by 8; and in the same manner, with the third and fourth; after which, he had 20 apples remaining for the fifth; how many apples did he distribute?

Ans. 80.

75. A could reap a field in 20 days, but if B assisted him for 6 days, he could reap it in 16 days; in how many days could B reap it alone? Ans. 30 days.

76. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which, being increased respectively, by 6 and 5, are in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$; required the numbers. Ans. 30 and 40.

77. When the price of a bushel of barley wanted but 3 cents to be to the price of a bushel of oats as 8 to 5, nine bushels of oats were received as an equivalent for 4 bushels of barley and 90 cents in money; what was the price of a bushel of each?

Ans. Oats 30 cts., and barley 45 cts.

78. Four places are situated in the order of the 4 letters, A, B, C, and D; the distance from A to D is 34 miles; the distance from A to B is to the distance from C to D, as 2 to 3; and $\frac{1}{4}$ the distance from A to B, added to $\frac{1}{2}$ the distance from C to D, is 3 times the distance from B to C. Required the respective distances.

Ans. A to B 12, B to C 4, and C to D 18 miles.

79. The ingredients of a loaf of bread are rice, flour, and water, and the weight of the whole is 15 pounds; the weight of the rice increased by 5 pounds, is $\frac{2}{3}$ the weight of the flour; and the weight of the water is $\frac{1}{5}$ the weight of the flour and rice together; what is the weight of each?

Ans. Rice 2℔, flour $10\frac{1}{2}$℔, and water $2\frac{1}{2}$℔.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

ART. **157.**—In order to find the value of any unknown quantity, it is evident, that we must obtain a single equation containing *it*, and known terms. Hence, when we have two or more equations, containing two or more unknown quantities, we must obtain from them a single equation containing only one unknown quantity. The method of doing this, is termed *elimination*, which may be briefly defined thus: *Elimination* is the process of deducing from two or more equations, containing two or more unknown quantities, a less number of equations containing one less unknown quantity

There are three methods of elimination.

1st. Elimination by substitution.

2d. Elimination by comparison.

3d. Elimination by addition and subtraction.

ELIMINATION BY SUBSTITUTION.

ART. **158.**—Elimination by substitution, consists in finding the value of one of the unknown quantities in one of the equations, in terms of the other unknown quantity and known terms, and substituting this, instead of the quantity, in the other equation.

To explain this, suppose we have the following equations, in which it is required to find the value of x and y.

NOTE.—The figures in the parentheses, are intended to number the equations for reference.

$$x+2y=17 \quad (1.)$$
$$2x+3y=28 \quad (2.)$$

By transposing $2y$ in the equation (1), we have $x=17-2y$. Substituting *this* value of x, instead of x in equation (2), we have

$$2(17-2y)+3y=28$$
$$\text{or,} \quad 34-4y+3y=28$$
$$\text{or,} \quad -y=28-34$$
$$y=6$$
$$\text{and} \quad x=17-2y=17-12=5.$$

Hence, when we have two equations, containing two unknown quantities, we have the following

RULE,

FOR ELIMINATION BY SUBSTITUTION.

Find an expression for the value of one of the unknown quantities in either equation, and substitute this value in place of the same unknown quantity in the other equation; there will thus be formed a new equation, containing only one unknown quantity.

NOTE.—In finding an expression for the value of one of the unknown quantities, let that be taken which is the least involved.

Find the values of the unknown quantities in each of the following equations.

1. $x+5y=38$. $3x+4y=37$. Ans. $x=3$. $y=7$.

2. $2x+4y=22$. $5x+7y=46$. Ans. $x=5$. $y=3$.

3. $3x+5y=57$. $5x+3y=47$. Ans. $x=4$. $y=9$.

4. $4x-3y=26$. $3x-4y=16$. Ans. $x=8$. $y=2$.

5. $2x-3y=-4$. $x-\frac{y}{3}=12$. Ans. $x=16$. $y=12$.

6. $x-y=10$. $\frac{x}{5}-\frac{y}{3}=0$. Ans. $x=25$. $y=15$.

7. $\frac{y}{5}-\frac{x}{4}=1$ $5x-3y=10$. Ans. $x=20$. $y=30$.

8. $\frac{2x}{7}-\frac{3y}{8}=0$. $\frac{2x}{3}+\frac{3y}{4}=26$. Ans. $x=21$. $y=16$.

ELIMINATION BY COMPARISON.

ART. **159.**—Elimination by comparison, consists in finding the value of the same unknown quantity in two different equations, and then placing these values equal to each other.

To illustrate this method, we will take the same equations which were used to explain elimination by substitution.

$$x+2y=17 \quad (1.)$$
$$2x+3y=28 \quad (2.)$$

By transposing $2y$ in equation (1), we have $x=17-2y$.

By transposing $3y$ in equation (2), and dividing by 2, we have

$$x=\frac{28-3y}{2}.$$

Placing these values of x equal to each other,

$$\frac{28-3y}{2}=17-2y$$
$$\text{or, } 28-3y=34-4y$$
$$\text{or, } y=6.$$

The value of x may be found in a similar manner, by first finding the values of y, and placing them equal to each other. But, after having found the value of one of the unknown quantities, the value of the other may be found most readily by substitution, as in the preceding article. Thus, $x=17-2y=17-12=5$.

REVIEW.—157. What is necessary in order to find the value of any unknown quantity? When we have two equations, containing two unknown quantities, what is necessary, in order to find the value of one of them? What is elimination? How many methods of elimination are there? 158. In what does elimination by substitution consist? What is the rule for elimination by substitution? 159. In what does elimination by comparison consist?

Hence, when we have two equations, containing two unknown quantities, we have the following

RULE,

FOR ELIMINATION BY COMPARISON.

Find an expression for the value of the same unknown quantity in each of the given equations, and place these values equal to each other; there will thus be formed a new equation, containing only one unknown quantity.

Find the value of each of the unknown quantities in the following equations, by the preceding rule.

1. $x+3y=16.$ $x+5y=22.$ Ans. $x=7.$ $y=3.$
2. $3x+5y=29.$ $3x-5y=19.$ Ans. $x=8.$ $y=1.$
3. $5x-2y=4.$ $2x-y=1.$ Ans. $x=2.$ $y=3.$
4. $\frac{x}{2}-\frac{y}{3}=2.$ $\frac{x}{3}-y=-1.$ Ans. $x=6.$ $y=3.$
5. $\frac{x}{9}-\frac{y}{8}=1.$ $\frac{x}{6}+\frac{y}{4}=12.$ Ans. $x=36.$ $y=24.$
6. $\frac{x}{4}-\frac{y}{4}=1.$ $\frac{x}{3}+\frac{y}{2}=8.$ Ans. $x=12.$ $y=8.$
7. $\frac{x}{5}+\frac{y}{2}=14.$ $\frac{x}{9}-\frac{y}{5}=3.$ Ans. $x=45.$ $y=10.$
8. $\frac{2x}{7}+\frac{3y}{5}=27.$ $\frac{3x}{9}-\frac{y}{7}=2.$ Ans. $x=21.$ $y=35.$
9. $\frac{3x}{2}+2y-x+\frac{2y}{3}=42.$ . . . . Ans. $x=20.$
   $3x-\frac{4y}{3}=40+\frac{x}{5}.$ . . . . $y=12.$
10. $\frac{3x-5y}{2}=\frac{2x+4}{7}-1.$ . . . . Ans. $x=12.$
    $6-\frac{x-2y}{4}=\frac{x+y}{3}.$ . . . . $y=6.$

ELIMINATION BY ADDITION AND SUBTRACTION.

ART. **160.**—Elimination by addition and subtraction, consists in multiplying or dividing two equations, so as to render the coëfficient of one of the unknown quantities, the same in both; and then, by adding or subtracting, to cause the term containing it to disappear.

To explain this method, we will take the same equations used to illustrate elimination by substitution and comparison.

$$x+2y=17 \quad (1.)$$
$$2x+3y=28 \quad (2.)$$

If we multiply equation (1) by 2, so as to make the coëfficient of x the same as in the second equation, we have

$$2x+4y=34 \quad (3.)$$
$$\underline{2x+3y=28}, \text{ equation (2) brought down.}$$
$$y=6$$

Since the coëfficient of x has the *same* sign in these equations, if we subtract, the terms containing x will cancel each other, and the resulting equation will contain only y, the value of which may then readily be found. After this, by substituting the value of y, as before, the value of x is easily obtained.

To illustrate the method of eliminating, when the coëfficients of the unknown quantity to be eliminated, have contrary signs in the two equations, suppose we have the following, in which it is required to eliminate y.

$$3x-5y=6 \quad (1.)$$
$$4x+3y=37 \quad (2.)$$

It is obvious, that if we multiply equation (1) by 3 and (2) by 5, that the coëfficients of y will be the same. Thus,

$$9x-15y=18$$
$$\underline{20x+15y=185}$$
$$\text{adding, } 29x \quad = \quad 203$$
$$x \quad = \quad 7.$$

Substituting this value of x in equation (2), we have

$$28+3y=37$$
$$3y=9$$
$$y=3$$

From this we see, that after making the coëfficients of the quantity to be eliminated, the same in both equations, if the signs are *alike*, we must subtract; but if they are *unlike*, we must *add* them.

Hence, when we have two equations, containing two unknown quantities, we have the following

RULE,

FOR ELIMINATION BY ADDITION AND SUBTRACTION.

Multiply, or divide the equations, if necessary, so that one of the unknown quantities will have the same coëfficient in both. Then take the difference, or the sum of the equations, according as the signs of the equal terms are alike or unlike, and the resulting equation will contain only one unknown quantity.

REMARK.—When the coëfficients of the unknown quantities to be eliminated are prime to each other, they may be equalized, by multiplying each

REVIEW.—159. What is the rule for elimination by comparison? 160. In what does elimination by addition and subtraction consist? What is the rule for elimination by addition and subtraction?

equation by the coëfficient of the unknown quantity in the other. When the coëfficients are not prime, find their least common multiple, and multiply each equation by the quotient obtained by dividing the least common multiple by the coëfficient of the unknown quantity to be eliminated in the other equation.

If the equations have fractional coëfficients, they ought to be cleared before applying the rule.

Find the value of the unknown quantities in each of the following equations, by the preceding rule.

1. $3x+2y=21.$ Ans. $x=5.$
   $x-2y=-1.$ $y=3.$
2. $3x-2y=7.$ Ans. $x=5.$
   $5y-2x=10.$ $y=4.$
3. $2x-y=3.$ Ans. $x=4.$
   $3x+2y=22.$ $y=5.$
4. $3x+2y=19$ Ans. $x=5.$
   $2x-3y=4.$ $y=2.$
5. $\frac{x}{4}+\frac{y}{5}=8.$ Ans. $x=20.$
   $\frac{x}{5}+\frac{y}{3}=9.$ $y=15.$
6. $\frac{x}{2}-\frac{y}{3}=3.$ Ans. $x=12.$
   $\frac{x}{6}+\frac{y}{9}=3.$ $y=9.$
7. $\frac{x+y}{5}-\frac{x-y}{2}=3.$ . . . . Ans. $x=4.$
   $\frac{x-y}{2}+\frac{x+y}{10}=0.$ . . . . $y=6.$

QUESTIONS PRODUCING EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

ART. **161.**—The questions contained in Art. 156, were all capable of being solved by using one unknown quantity; although, several of the examples contained two, and in some cases more, unknown quantities. In those questions, however, there was such a connection existing between the several quantities, that it was easy to express each one in terms of the other. But it frequently happens, that in a problem containing two unknown quantities, there may be no direct relation existing between them, by means of which either of them may be found in terms of the other. In such a case, it becomes necessary to use a separate symbol for each unknown quantity, and then to find the equations containing these symbols, on the same principle as where there was but one unknown quantity; that is, in brief, *regard the symbols as the answer to the question, and then proceed in the same manner as it would be necessary to do, to prove the answer.* After the equations are obtained, the values of the unknown quantities may be found, by either of the three different modes of elimination.

We shall first give two examples, which can be solved by using either one or two unknown quantities.

In general, no more symbols should be used, than are really necessary; unless, by using them, the solution is rendered more simple.

1. Given, the sum of two numbers equal to 25, and their difference equal to 9, to find the numbers.

Solution, by using one unknown quantity.

Let $x=$ the less number; then $x+9=$ the greater.

And $x+x+9=25$.

$2x=16$

$x=8$, the less number; and $x+9=17$, the greater.

Solution, by using two unknown quantities.

Let $x=$ the greater, and $y=$ the less.

Then $x+y=25$ (1.)

And $x-y=\ 9$ (2.)

$2x=34$, by adding the two equations together.

$x=17$, the greater number.

$2y=16$, by subtracting equation (2) from equation (1).

$y=\ 8$, the less number.

2. The sum of two numbers is 44, and they are to each other as 5 to 6; required the numbers.

Solution, by using one unknown quantity.

Let $5x=$ the less number; then $6x=$ the greater.

And $5x+6x=44$.

$11x=44$

$x=4$

$5x=20$, the less number.

$6x=24$, the greater number.

Solution, by using two unknown quantities.

Let $x=$ the less number, and $y=$ the greater.

Then $x+y=44$ (1.)

And $x:y::5:6$

or, $6x=5y$ (2.) by multiplying means and extremes.

$6x+6y=264$ (3.) by multiplying equation (1) by 6.

$6y=264-5y$, by subtracting equation (2) from (3).

$11y=264$

$y=24$ and $x=44-y=20$.

Several of the following questions may also be solved by using only one unknown quantity.

3. There is a certain number consisting of two places of figures; the sum of the figures is equal to 6, and, if from the double of the

REVIEW.—161. In solving questions, when does it become necessary to use a separate symbol for each unknown quantity? How are the equations formed, from which the values of the unknown quantities are to be obtained?

number 6 be subtracted, the remainder is a number whose digits are those of the former in an inverted order; required the number?

In solving questions of this kind, the pupil must be reminded, that any number consisting of two places of figures, is equal to 10 times the figure in the ten's place, plus the figure in the unit's place. Thus, 23 is equal to $10\times2+3$. In a similar manner, 325 is equal to $100\times3+10\times2+5$.

Let $x=$ the digit in the place of tens, and $y=$ that in the place of units.

Then $10x+y=$ the number.

And $10y+x=$ the number, with the digits inverted.

Then $x+y=6$ (1.)

And $2(10x+y)-6=10y+x$ (2.)

or, $20x+2y-6=10y+x$.

$19x=8y+6$

$8x=-8y+48$, from equation (1), by multiplying by 8, and transposing.

$27x=54$, by adding.

$x=2$

$y=6-2=4$. Ans. 24.

4. What two numbers are those, to which if 5 be added, the sums will be to each other as 5 to 6; but, if 5 be subtracted from each, the remainders will be to each other as 3 to 4?

By the conditions of the question, we have the following proportions:

$$x+5 : y+5 :: 5 : 6$$
$$x-5 : y-5 :: 3 : 4.$$

Since, in every proportion, the product of the means is equal to the product of the extremes, we have the two equations

$$6(x+5)=5(y+5)$$
$$4(x-5)=3(y-5)$$

From these equations, the values of x and y are readily found to be 20 and 25.

REMARK.—Instead of saying, that the two sums will be to each other as 5 to 6, it will be the same to say, that the quotient of the second divided by the first, is equal to $\frac{6}{5}$, since 6 divided by 5, expresses the ratio of 5 to 6. This would give the following equations:

$$\frac{y+5}{x+5}=\frac{6}{5} \text{ and } \frac{y-5}{x-5}=\frac{4}{3}$$

which may be readily obtained from those given above.

NOTE.—In solving the following questions, after finding the equations, the values of the unknown quantities may be found by either of the three methods of elimination.

5. A grocer sold to one person 5 pounds of coffee and 3 pounds of sugar, for 79 cents; and to another, at the same prices, 3 pounds of coffee and 5 pounds of sugar, for 73 cents; what was the price of a pound of each? Ans. Coffee 11 cts., sugar 8 cts.

6. A farmer sold to one person 9 horses and 7 cows, for 300 dollars; and to another, at the same prices, 6 horses and 13 cows, for the same sum; what was the price of each?

Ans. Horses \$24, and cows \$12 each.

7. A vintner sold at one time, 20 dozen of port wine and 30 of sherry, and for the whole received 120 dollars; and, at another, 30 dozen of port and 25 of sherry, at the same prices as before, for 140 dollars; what was the price of a dozen of each sort of wine? Ans. Port \$3, and sherry \$2 per doz.

8. It is required to find two numbers, such that $\frac{1}{2}$ of the first and $\frac{1}{3}$ of the second shall be 22, and $\frac{1}{4}$ of the first and $\frac{1}{5}$ of the second shall be 12. Ans. 24 and 30.

9. If the greater of two numbers be added to $\frac{1}{3}$ of the less, the sum will be 37; but if the less be diminished by $\frac{1}{4}$ of the greater, the difference will be 20; what are the numbers? Ans. 28 and 27.

10. What two numbers are those, such that $\frac{1}{2}$ of the first diminished by $\frac{1}{3}$ of the second, shall be 5, and $\frac{1}{4}$ of the first diminished by $\frac{1}{5}$ of the second, shall be 2? Ans. 20 and 15.

11. A farmer has 2 horses, and a saddle worth 25 dollars; now, if the saddle be put on the first horse, his value will be double that of the second; but, if the saddle be put on the second horse, his value will be three times that of the first. Required the value of each horse. Ans. First \$15, second \$20.

12. A and B are in trade together with different sums; if 50 dollars be added to A's property, and 20 dollars taken from B's, they will have the same sum; and if A's property was 3 times, and B's 5 times as great as each really is, they would together have 2350 dollars; how much has each? Ans. A \$250, B. \$320.

13. A has two vessels containing wine, and finds, that $\frac{2}{5}$ of the first contains 96 gallons less than $\frac{3}{4}$ of the second; and that $\frac{5}{8}$ of the second contains as much as $\frac{4}{9}$ of the first; how much does each vessel hold? Ans. 720 and 512 galls.

14. There is a number consisting of two digits, which, divided by their sum, gives a quotient, 7; but if the digits be written in an inverse order, and the number so arising, be divided by their sum increased by 4, the quotient will be 3. Required the number.

Ans. 84.

15. If we add 8 to the numerator of a certain fraction, its value becomes 2; and if we subtract 5 from the denominator, its value becomes 3; required the fraction. Ans. $\frac{6}{7}$

16. If to the ages of A and B 18 be added, the result will be double the age of A; but, if from their difference 6 be subtracted, the result will be the age of B; required their ages.

Ans. A 30, B 12 yrs.

17. There are two numbers whose sum is 37, and if 3 times the less be subtracted from 4 times the greater, and the difference divided by 6, the quotient will be 6; what are the numbers?

Ans. 16 and 21.

18. It is required to find a fraction, such that if 3 be subtracted from the numerator and denominator, the value will be $\frac{1}{4}$; and if 5 be added to the numerator and denominator, the value will be $\frac{1}{2}$.

Ans. $\frac{7}{19}$.

19. A father gave his two sons, A and B, together 2400 dollars, to engage in trade; at the close of the year, A has lost $\frac{1}{4}$ of his capital, while B, having gained a sum equal to $\frac{1}{4}$ of his capital, finds that his money is just equal to that of his brother; what was the sum given by the father to each? Ans. A \$1500, B \$900.

20. If from the greater of two numbers 1 be subtracted, the remainder will be equal to 4 times the less; but, if to the less 3 be added, the sum will be $\frac{1}{3}$ of the greater; required the numbers.

Ans. 8 and 33.

21. A said to B, "Give me 100 dollars, and then I shall have as much as you." B said to A, "Give me 100 dollars, and then I shall have twice as much as you." How many dollars had each?

Ans. A \$500, B \$700.

22. If the greater of two numbers be multiplied by 5, and the less by 7, the sum of their products is 198; but if the greater be divided by 5, and the less by 7, the sum of their quotients is 6; what are the numbers? Ans. 20 and 14.

23. Seven years ago the age of A was just three times that of B; and seven years hence, A's age will be just double that of B; what are their ages? Ans. A's 49, B's 21 yrs.

24. There is a certain number consisting of two places of figures, which being divided by the sum of its digits, the quotient is 4, and if 27 be added to it, the digits will be inverted; required the number. Ans. 36.

25. A grocer has two kinds of sugar, of such quality that one pound of each are together worth 20 cents; but if 3 pounds of the first, and 5 pounds of the second kind be mixed, a pound of the mixture will be worth 11 cents; what is the value of a pound of each sort? Ans. 6 cts., and 14 cts.

26. A boy lays out 84 cents for lemons and oranges, giving 3 cents a piece for the lemons, and 5 cents a piece for the oranges; he afterward sold $\frac{1}{2}$ of the lemons and $\frac{1}{3}$ of the oranges, for 40

cents, and by so doing cleared 8 cents on what he sold; what number of each did he purchase?

Ans. 8 lemons and 12 oranges.

27. A person spends 30 cents for peaches and apples, buying his peaches at 4, and his apples at 5 for a cent; he afterward sells $\frac{1}{2}$ of his peaches, and $\frac{1}{3}$ of his apples, at the same rate he bought them, for 13 cents; how many of each did he buy?

Ans. 72 peaches and 60 apples.

28. A owes 500 dollars, and B owes 600 dollars, but neither has sufficient money to pay his debts. A said to B, "Lend me $\frac{1}{5}$ of your money, and I shall have enough to discharge my debts." B said to A, "Lend me $\frac{1}{4}$ of your money, and I can pay mine." How much money has each? Ans. A $400, B $500.

29. A merchant bought two pieces of cloth for 236 dollars, the first piece at 4, and the second at 7 dollars per yard; but the cloth getting damaged, he sold $\frac{7}{8}$ of the first piece, and $\frac{3}{5}$ of the second, for 160 dollars, by which he lost 8 dollars on what he sold; what was the number of yards in each piece?

Ans. 24 yards in the first, and 20 yards in the second.

30. A son said to his father, "How old are we?" The father replied, "Six years ago my age was $3\frac{1}{3}$ times yours, but 3 years hence, my age will be only $2\frac{1}{6}$ times yours." Required the age of each. Ans. Father's age 36, son's 15 yrs.

31. A person has two horses, and two saddles, one of which cost 50, and the other 2 dollars. If he places the best saddle upon the first horse, and the other on the second, then the latter is worth 8 dollars less than the former; but if he puts the worst saddle upon the first, and the best upon the second horse, then the value of the latter is to that of the former as 15 to 4. Required the value of each horse. Ans. First $30, second $70.

32. A farmer having mixed a certain number of bushels of oats and rye, found, that if he had mixed 6 bushels more of each, he would have mixed 7 bushels of oats for every 6 of rye; but if he had mixed 6 bushels less of each, he would have put in 6 bushels of oats for every 5 of rye. How many bushels of each did he mix? Ans. Oats 78, rye 66 bu.

33. A person having laid out a rectangular yard, observed, that if each side had been 4 yards longer, the length would have been to the breadth, as 5 to 4; but, if each had been 4 yards shorter, the length would have been to the breadth, as 4 to 3; required the length of the sides. Ans. Length, 36, breadth 28 yards.

34. A farmer rents a farm for 245 dollars per annum; the tillable land being valued at 2 dollars an acre, and the pasture at 1 dollar and 40 cents an acre; now the number of acres tillable, is

to the excess of the tillable above the pasture, as 14 to 9; how many were there of each? A. Tillable 98, pasture 35 acres.

35. Two shepherds, A and B, are intrusted with the charge of two flocks of sheep; at the end of the first year, it is found, that A's flock has increased 10, and B's diminished 20, when their numbers are to each other, as 4 to 3; during the second year, A's flock loses 20, and B's gains 10, when their numbers are to each other as 6 to 7. Required the number in each flock at first.

Ans. A's had 70, and B's 80 sheep.

36. After drawing 15 gallons from each of 2 casks of wine, the quantity remaining in the first, is $\frac{2}{3}$ of that in the second; after drawing 25 gallons more from each, the quantity left in the first, is only half that in the second. Required the number of gallons in each before the first drawing. Ans. 65 and 90 galls.

37. There is a fraction, such that if 1 be added to the numerator, and the numerator to the denominator, its value will be $\frac{1}{4}$; but if the denominator be increased by unity, and the numerator by the denominator, its value will be $\frac{8}{7}$; find it. Ans. $\frac{3}{13}$.

38. Find two numbers in the ratio of 5 to 7, to which two other required numbers, in the ratio of 3 to 5, being respectively added, the sums shall be in the ratio of 9 to 13, and the difference of their sums equal to 16. Ans. 30 and 42, 6 and 10.

Let the first two numbers be represented by $5x$ and $7x$, and the other two by $3y$ and $5y$.

39. A farmer, with 28 bushels of barley, worth 28 cents per bushel, would mix rye at 36 cents, and wheat at 48 cents per bushel, so that the whole mixture may consist of 100 bushels, and be worth 40 cents a bushel; how many bushels of rye, and how many of wheat must be mixed with the barley?

Ans. Rye 20, and wheat 52 bu

40. Two loaded wagons were weighed, and their weights were found to be in the ratio of 4 to 5; part of their loads, which were in the ratio of 6 to 7, being taken out, their weights were then found to be in the ratio of 2 to 3, and the sum of their weights was then 10 tons; what were their weights at first?

Ans. 16 and 20 tons.

41. A person had two casks and a certain quantity of wine in each; in order to have the same quantity in each cask, he poured as much out of the first cask into the second as it already contained; he next poured as much out of the second, into the first, as it then contained; and lastly, he poured out as much from the first into the second, as there was remaining in it; after this, he had 16 gallons in each cask; how many gallons did each contain at first? Ans. First 22, and second 10 galls.

SIMPLE EQUATIONS, CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

ART. **162.**—Equations involving three or more unknown quantities may be solved, by either of the three methods of elimination explained in the preceding Articles, as we shall now proceed to show, by solving an example by each of these methods.

Suppose we have the three following equations, in which it is required to find the values of x, y, and z.

$$x+2y+z=20 \quad (1.)$$
$$2x+y+3z=31 \quad (2.)$$
$$3x+4y+2z=44 \quad (3.)$$

Solution by substitution.

From equation (1), $x=20-2y-z$.

Substituting this in equation (2), we have

$$2(20-2y-z)+y+3z=31.$$
$$\text{or, } 40-4y-2z+y+3z=31.$$
$$3y-z=9 \quad (4.)$$

Substituting the same value of x in equation (3), we have

$$3(20-2y-z)+4y+2z=44.$$
$$\text{or, } 60-6y-3z+4y+2z=44.$$
$$2y+z=16 \quad (5.)$$
$$3y-z=9 \quad (4.)$$

Here the values of y and z are readily found by the rule, Art 158, to be 5 and 6; then substituting these values in equation (1), we find $x=4$.

Solution by comparison.

From equation (1), $x=20-2y-z$

" " (2), $x=\dfrac{31-y-3z}{2}$

" " (3), $x=\dfrac{44-4y-2z}{3}$

Comparing the first and second values of x, we have

$$20-2y-z=\frac{31-y-3z}{2}$$
$$\text{or, } 40-4y-2z=31-y-3z$$
$$\text{or, } 3y-z=9 \quad (4.)$$

Comparing the first and third values of x, we have

$$20-2y-z=\frac{44-4y-2z}{3}$$
$$\text{or, } 60-6y-3z=44-4y-2z$$
$$2y+z=16 \quad (5.)$$

From equations (4) and (5), the values of y and z, and then x, may be found by the rule, Art. 159.

Solution by addition and subtraction.

Multiplying equation (1) by 2, to render the coëfficient of x the same as in equation (2), we have

$$\begin{array}{lrl} & 2x+4y+2z=40 & \\ \text{equation (2) is} & 2x+\ \ y+3z=31 & \\ \hline \text{by subtracting,} & 3y-\ z=\ 9 & (4.) \end{array}$$

Next, multiplying equation (1) by 3, to render the coëfficient of x the same as in equation (3), we have

$$\begin{array}{lrl} & 3x+6y+3z=60 & \\ \text{equation (3) is} & 3x+4y+2z=44 & \\ \hline \text{by subtracting,} & 2y+\ z=16 & (5.) \\ & 3y-\ z=\ 9 & (4.) \\ \hline \text{by adding,} & 5y\ =\ 25 & \\ & y\ =\ \ 5 & \end{array}$$

Then $10+z=16$, and $z=6$.

And $x+10+6=20$, and $x=4$.

REMARK.—The methods of elimination by substitution and comparison, when there are more than two unknown quantities, are merely an extension of the rules already presented, in Articles 158 and 159; therefore, it is unnecessary to repeat them here. When the number of unknown quantities is three or more, and particularly when each of the unknown quantities is found in all the equations, the method of elimination by addition and subtraction is generally preferred; we shall, therefore, illustrate it by another example.

Let it be required to find the value of each of the unknown quantities in the following equations.

$$\begin{array}{rl} v+2x+3y+4z=30 & (1.) \\ 2v+3x+\ y+\ z=15 & (2.) \\ 3v+\ x+2y+3z=23 & (3.) \\ 4v+2x-y+14z=61 & (4.) \end{array}$$

Let us first eliminate v: this may be done by making the coëfficient of v, in one of the equations, the same as in the other three, and then subtracting.

$2v+4x+6y+8z=60$, by multiplying equation (1) by 2.

$$\begin{array}{rl} 2v+3x+y+z=15 & (2.) \\ \hline x+5y+7z=45 & (5.)\text{, by subtracting.} \end{array}$$

$3v+6x+9y+12z=90$, by multiplying equation (1) by 3.

$$\begin{array}{rl} 3v+x+2y+3z=23 & (3.) \\ \hline 5x+7y+9z=67 & (6.)\text{, by subtracting.} \end{array}$$

$4v+8x+12y+16z=120$, by multiplying equation (1) by 4.

$$\begin{array}{rl} 4v+2x-y+14z=\ 61 & (4.) \\ \hline 6x+13y+2z=\ 59 & (7.)\text{, by subtracting.} \end{array}$$

Collecting into one place, the new equations (5), (6), and (7), we find, that the number of unknown quantities, as well as the number of equations, is *one* less.

$$x+5y+7z=45 \quad (5.)$$
$$5x+7y+9z=67 \quad (6.)$$
$$6x+13y+2z=59 \quad (7.)$$

The next step is to eliminate x, by making the coëfficient of x, in one of the equations, the same as in each of the others, and then subtracting.

$5x+25y+35z=225$, by multiplying equation (5) by 5.

$$\begin{array}{r} 5x+7y+9z=\ 67 \\ \hline 18y+26z=158 \quad (8.) \end{array}$$

$6x+30y+42z=270$, by multiplying equation (5) by 6.

$$\begin{array}{r} 6x+13y+2z=\ 59 \\ \hline 17y+40z=211 \quad (9.) \end{array}$$

Bringing together equations (8) and (9), we find, that the number of equations, as well as of unknown quantities, is now *two* less.

$$18y+26z=158 \quad (8.)$$
$$17y+40z=211 \quad (9.)$$

$306y+442z=2686$, by multiplying equation (8) by 17.

$306y+720z=3798$, by multiplying equation (9) by 18.

$$\begin{array}{r} \hline 278z=1112 \\ z=\quad 4 \end{array}$$

Substituting the value of z, in equation (9), we get

$$\begin{array}{r} 17y+160=211 \\ 17y=\ 51 \\ y=\ \ 3. \end{array}$$

Substituting the values of y and z, in equation (5), we get

$$x+15+28=45$$
$$x=2$$

And lastly, substituting the values of x, y, and z, in equation (1), we get

$$v+4+9+16=30$$
$$\text{or, } v=1.$$

From the preceding example, we derive the

GENERAL RULE,

FOR ELIMINATION BY ADDITION AND SUBTRACTION.

1st. *Combine any one of the equations with each of the others, so as to eliminate the same unknown quantity; there will thus arise a new class of equations, containing one less unknown quantity.*

2d. *Combine any one of these new equations with each of the others, so as to eliminate another unknown quantity; there will thus arise another class of equations, containing two less unknown quantities.*

3d. *Continue this series of operations until a single equation is obtained, containing but one unknown quantity, from which its value may be easily found; then, by going back, and substituting this value in the derived equations, the values of the other unknown quantities may be readily found.*

REMARK.—When the number of unknown quantities in each equation, is less than the whole number of unknown quantities involved, the method of substitution will generally be found the shortest. By solving several of the following examples, by each of the three different methods, the pupil will be able to appreciate their relative excellence in different cases.

EXAMPLES,

TO BE SOLVED BY EITHER OF THE DIFFERENT METHODS OF ELIMINATION.

1. $\left.\begin{array}{l} x+y=50. \\ x+z=28. \\ y+z=42. \end{array}\right\}$ . . . . Ans. $x=18$. $y=32$. $z=10$.

2. $\left.\begin{array}{l} 3x+5y=\ 76. \\ 4x+6z=108. \\ 5z+7y=106. \end{array}\right\}$ . . . . Ans. $x=12$. $y=8$. $z=10$.

3. $\left.\begin{array}{l} x+y+z=26. \\ x+y-z=-6. \\ x-y+z=12. \end{array}\right\}$ . . . . Ans. $x=3$. $y=7$. $z=16$.

4. $x+\frac{y}{2}=100;\ y+\frac{z}{3}=100;\ z+\frac{x}{4}=100.$ . . . $\left\{\begin{array}{l} \text{Ans. } x=64. \\ y=72. \\ z=84. \end{array}\right.$

5. $\left.\begin{array}{l} 2x-y+z=9. \\ x-2y+3z=14. \\ 3x+4y-2z=7. \end{array}\right\}$ . . . . Ans. $x=3$. $y=2$. $z=5$.

6. $\left.\begin{array}{l} 2x-3y+5z=15. \\ 3x+2y-z=8. \\ -x+5y+2z=21. \end{array}\right\}$ . . . . Ans. $x=2$. $y=3$. $z=4$.

7. $\left.\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{7}=22. \\ \frac{x}{3}+\frac{y}{5}+\frac{z}{2}=31. \\ \frac{x}{4}+\frac{y}{2}+\frac{z}{3}=32. \end{array}\right\}$ . . . . Ans. $x=12$. $y=30$. $z=42$.

8. $\left.\begin{array}{l} \frac{x}{3}-\frac{y}{2}+z=3. \\ \frac{x}{6}+\frac{y}{4}-\frac{z}{3}=1. \\ \frac{x}{2}-\frac{y}{4}+z=5. \end{array}\right\}$ . . . . Ans. $x=6$ $y=4$. $z=3$.

QUESTIONS PRODUCING EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

ART. **163.**—When a question contains *three* or more unknown quantities, equations involving them, can be found on the same principle as in questions containing *one* or *two* unknown quantities. (See Articles 156 and 161.) The values of the unknown quantities may then be found by either of the three methods of elimination.

REMARK.—The method of elimination to be preferred, will depend on the manner in which the unknown quantities are combined, and must be left to the judgment of the pupil. When such a relation exists between the different unknown quantities, that one or more of them can be expressed directly in terms of another, it should be done, as this generally renders the solution more simple.

1. A person has 3 ingots, composed of 3 different metals in different proportions; a pound of the first contains 7 ounces of silver, 3 of copper, and 6 of tin; a pound of the second consists of 12 ounces of silver, 3 of copper, and 1 of tin; and a pound of the third, of 4 ounces of silver, 7 of copper, and 5 of tin. How much of each of the ingots must be taken, to form another ingot of 1 pound weight, consisting of 8 ounces of silver, $3\frac{3}{4}$ of copper, and $4\frac{1}{4}$ of tin?

Let x, y, z, be the number of ounces to be taken of the 3 ingots respectively.

Then, since 16 ounces of the first contain 7 ounces of silver, 1 ounce will contain $\frac{7}{16}$ of an ounce of silver; and hence, x ounces will contain $\frac{7x}{16}$ ounces of silver.

In the same manner, y ounces of the second will contain $\frac{12y}{16}$ ounces of silver; and z ounces of the third will contain $\frac{4z}{16}$ ounces of silver. But, by the question, the number of ounces of silver in a pound of the new ingot, is to be 8, hence

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8.$$

Or, by clearing it of fractions,

$$7x+12y+4z=128 \quad (1.)$$

REVIEW.—162. What is the general rule for elimination by addition and subtraction? When is the method of elimination by substitution to be preferred to this? 163. Upon what principle are equations formed, when a question contains three or more unknown quantities? When should we use a less number of symbols than there are unknown quantities?

Reasoning in a similar manner with reference to the copper and the tin, we have the two following equations:

$$3x+3y+7z=60 \quad (2.)$$
$$6x+\ y+5z=68 \quad (3.)$$

The coëfficient of y being the simplest, will be most easily eliminated.

If we multiply the second equation by 4, and take the first equation from the product, the result is

$$5x+24z=112 \quad (4.)$$

If we multiply the third equation by 3, and take the second from the product, the result is

$$15x+8z=144 \quad (5.)$$

If we multiply the last equation by 3, and take the preceding equation from it, the result is

$$40x=320$$
$$x=8$$

Substituting this value of x in equation (5), we have

$$120+8z=144$$
$$z=3$$

And substituting these values of x and z, in equation (3),

$$48+y+15=68$$
$$y=5$$

Hence, the new ingot will contain 8 ounces of the first, 5 of the second, and 3 of the third.

2. The sums of three numbers, taken two and two, are 27, 32, and 35; required the numbers. Ans. 12, 15, and 20.

3. The sum of three numbers is 59; $\frac{1}{2}$ the difference of the first and second is 5, and $\frac{1}{2}$ the difference of the first and third is 9; required the numbers. Ans. 29, 19, and 11.

4. There are three numbers, such that the first, with $\frac{1}{2}$ the second, is equal to 14; the second, with $\frac{1}{3}$ part of the third, is equal to 18; and the third, with $\frac{1}{4}$ part of the first, is equal to 20; required the numbers. Ans. 8, 12, and 18.

5. A person bought three silver watches; the price of the first, with $\frac{1}{2}$ the price of the other two, was 25 dollars; the price of the second, with $\frac{1}{3}$ of the price of the other two, was 26 dollars; and the price of the third, with $\frac{1}{2}$ the price of the other two, was 29 dollars; required the price of each. A. \$8, \$18, and \$16.

6. Find three numbers, such that the first with $\frac{1}{3}$ of the other two, the second with $\frac{1}{4}$ of the other two, and the third with $\frac{1}{5}$ of the other two, shall each be equal to 25. Ans. 13, 17, and 19.

7. A boy bought at one time 2 apples and 5 pears, for 12 cents; at another, 3 pears and 4 peaches, for 18 cents; at another, 4 pears

and 5 oranges, for 28 cents; and at another, 5 peaches and 6 oranges, for 39 cents; required the cost of each kind of fruit.

Ans. Apples 1 cent, pears 2, peaches, 3, oranges 4 cts., each.

8. A and B together possess only $\frac{2}{3}$ as much money as C; B and C together, have 6 times as much as A; and B has 680 dollars less than A and C together; how much has each?

Ans. A \$200, B \$360, and C \$840.

9. A, B, and C together, have 1820 dollars; if B give A 200 dollars, then A will have 160 dollars more than B; but if B receive 70 dollars from C, they will both have the same sum; how much has each? Ans. A \$400, B \$640, and C \$780.

10. Three persons, A, B, and C, compare their money; A says to B, "Give me 700 dollars, and I shall have twice as much as you will have left." B says to C, "Give me 1400 dollars, and I shall have three times as much as you will have left." And C says to A, "Give me 420 dollars, and then I shall have five times as much as you will have left." How much has each?

Ans. A \$980, B \$1540, and C \$2380.

11. A certain number is expressed by three figures, and the sum of the figures is 11; the figure in the place of units, is double that in the place of hundreds; and if 297 be added to the number, its figures will be inverted; required the number. Ans. 326.

12. Three persons, A, B, and C, together, have 2000 dollars; if A gives B 200 dollars, then B will have 100 dollars more than C; but, if B gives A 100 dollars, then B will have only $\frac{3}{4}$ as much as C; required the sum possessed by each.

Ans. A \$500, B \$700, and C \$800.

13. There are three numbers whose sum is 83; if, from the first and second you subtract 7, the remainders are as 5 to 3; but if from the second and third, you subtract 3, the remainders are to each other as 11 to 9; required the numbers. A. 37, 25, 21.

14. Divide 180 dollars between three persons, A, B, and C, so that twice A's share plus 80 dollars, three times B's share, plus 40 dollars, and four times C's share plus 20 dollars, may be all equal to each other. Ans. A \$70, B \$60, and C \$50.

15. There are three numbers whose sum is 78; $\frac{1}{3}$ of the first is to $\frac{1}{4}$ of the second, as 1 to 2; also, $\frac{1}{4}$ of the second is to $\frac{1}{5}$ of the third, as 2 to 3; what are the numbers? Ans. 9, 24, and 45.

16. A, B, and C, have a sum of money; A's share exceeds $\frac{4}{7}$ of the shares of B and C, by 30 dollars; B's share exceeds $\frac{3}{8}$ of the shares of A and C, by 30 dollars; and C's share exceeds $\frac{2}{9}$ of the shares of A and B, by 30 dollars; what is the share of each?

Ans. A's \$150, B's \$120, and C's \$90.

17. If A and B can perform a certain work in 12 days, A and C in 15 days, and B and C in 20 days, in what time could each do it alone? Ans. A 20, B 30, and C 60 days

18. A number, expressed by three figures, when divided by the sum of the figures plus 9, gives a quotient 19; also, the middle figure is equal to half the sum of the first and third; and, if 198 be added to the number, we obtain a number with the same figures in an inverted order; what is the number? Ans. 456.

19. A farmer mixes barley at 28 cents, with rye at 36, and wheat at 48 cents per bushel, so that the whole is 100 bushels, and worth 40 cents per bushel. Had he put twice as much rye, and 10 bushels more of wheat, the whole would have been worth exactly the same per bushel; how much of each kind was there?
Ans. Barley 28, rye 20, and wheat 52 bushels.

20. A, B, and C, in a hunting excursion, killed 96 birds, which they wish to share equally; in order to do this, A, who has the most, gives to B and C as many as they already had; next, B gives to A and C as many as they had after the first division; and lastly, C gives to A and B as many as they both had after the second division; it was then found, that each had the same number; how many had each at first? Ans. A 52, B 28, and C 16.

CHAPTER V.

SUPPLEMENT TO EQUATIONS OF THE FIRST DEGREE.

GENERALIZATION.

Art. **164.**—Equations are termed *literal*, when the known quantities are represented, either entirely or partly, by letters. Quantities represented by letters, are termed *general* values—because, by giving particular values to the letters, the solution of one problem, furnishes a *general* solution to all others of the same kind.

The answer to a problem, when the known quantities are represented by letters, is termed a *formula;* and a formula, expressed in ordinary language, furnishes a *rule.*

By the application of Algebra to the solution of general questions, a great number of useful and interesting truths and rules may be established. We shall now proceed to illustrate this subject, by a few examples.

Art. **165.**—1. Let it be required to find a number, which being divided by 3, and by 5, the sum of the quotients will be 16.

Let $x=$ the number; then $\frac{x}{3}+\frac{x}{5}=16$.

$$5x+3x=16\times15$$
$$8x=16\times15$$
$$x=2\times15=30.$$

2 Again, let it be required to find another number, which being divided by 4, and by 7, the sum of the quotients will be 11.

By proceeding, as in the preceding question, we find the number to be 28.

Instead, however, of solving every example of the same kind separately, we may give a general solution, that will embrace all the particular questions. Thus:

3. Let it be required to find a number, which being divided by two given numbers, a and b, the sum of the quotients may be equal to another given number, c.

Let $x=$ the number; then $\frac{x}{a}+\frac{x}{b}=c$.

$$bx+ax=abc$$
$$(a+b)x=abc$$
$$x=\frac{abc}{a+b}.$$

The answer to this question is termed a formula; it shows, that the required number is equal to the continued product of a, b, and c, divided by the sum of a and b. Or, it may be expressed in ordinary language, thus: *Multiply together the three given numbers, and divide the product by the sum of the divisors; the result will be the required number.*

The pupil may test the accuracy of this rule, by solving the following examples, and verifying the results.

4. Find a number, which being divided by 3, and by 7, the sum of the quotients may be 20. Ans. 42.

5. Find a number, which being divided by $\frac{1}{3}$ and $\frac{1}{4}$, the sum of the quotients may be 1. Ans. $\frac{1}{7}$.

ART. **166.**—1. The sum of 500 dollars is to be divided between two persons, A and B, so that A may have 50 dollars less than B.

Ans. A \$225, B \$275.

To make this question general, let it be stated as follows:

REVIEW.—164. When are equations termed literal? When are quantities termed general? When is the answer to a problem termed a formula? What is a formula called, when expressed in ordinary language? 165. Example 3. What is the answer to this question, expressed in ordinary language?

2. To divide a given number, a, into two such parts, that their difference shall be b. Or thus:

The sum of two numbers is a, and their difference b; required the numbers.

Let $x=$ the greater number, and $y=$ the less.

Then $x+y=a$

And $x-y=b$

By addition, $2x=a+b$

$$x=\frac{a+b}{2}=\frac{a}{2}+\frac{b}{2}$$

By subtraction, $2y=a-b$

$$y=\frac{a-b}{2}=\frac{a}{2}-\frac{b}{2}.$$

This formula, when expressed in ordinary language, gives the

RULE,

FOR FINDING TWO QUANTITIES, WHEN THEIR SUM AND DIFFERENCE ARE GIVEN.

To find the greater, add half the difference to half the sum. To find the less, subtract half the difference from half the sum.

Let the learner test the accuracy of the rule, by finding two numbers, such that their sum shall be equal to the first number in each of the following examples, and their difference equal to the second.

3. Sum 200, difference 50. . . . . Ans. 125, 75.
4. Sum 100, difference 25. . . . . Ans. $62\frac{1}{2}$, $37\frac{1}{2}$.
5. Sum 15, difference 10. . . . . Ans. $12\frac{1}{2}$, $2\frac{1}{2}$.
6. Sum $5\frac{1}{2}$, difference $\frac{3}{4}$. . . . . Ans. $3\frac{1}{8}$, $2\frac{3}{8}$.

ART. **167.**—1. A can perform a certain piece of work in 3 days, and B in 4 days; in what time can they both together do it?

Ans. $1\frac{5}{7}$ days.

To make this question general, let it be stated thus:

2. A can perform a certain piece of work in m days, and B can do it in n days; in how many days can they both together do it?

Let $x=$ the number of days in which they can both do it.

Then $\frac{1}{x}=$ the part of the work which both can do in one day.

Also, if A can do the work in m days, he can do $\frac{1}{m}$ part of it in one day. And, if B can do the work in n days, he can do $\frac{1}{n}$ part of it in one day. Hence, the part of the work which both can do in one day, is represented by $\frac{1}{m}+\frac{1}{n}$, and also by $\frac{1}{x}$.

Therefore, ... $\frac{1}{m}+\frac{1}{n}=\frac{1}{x}$.

$$nx+mx=mn$$

$$x=\frac{mn}{m+n}.$$

This result, expressed in ordinary language, gives the following

RULE.

Divide the product of the numbers expressing the time in which each can perform the work by their sum; the quotient will be the time in which they can jointly perform it.

The question can be made more general, by expressing it thus: An agent, A, can produce a certain effect, e, in a time, t; another agent, B, can produce the same effect, in a time, t'; in what time can they both do it? Both the result and the rule would be the same as that already given.

The following examples will illustrate the rule.

3. A cistern is filled by one pipe in 6, and by another in 9 hours; in what time will it be filled by both together? A. $3\frac{3}{5}$ hrs.

4. One man can drink a keg of cider in 5 days, and another in 7 days; in what time can both together drink it? A. $2\frac{11}{12}$ dys.

ART. **168.**—Let it be required to find a rule for dividing the gain or loss in a partnership, or, as it is generally termed, fellowship.

First, take a particular question.

1. A, B, and C, engage in trade, and put in stock in the following proportions: A put in 3 dollars, as often as B put in 4, and as often as C put in 5 dollars. Their gains amounted to 60 dollars; required the share of each, the gains being divided in proportion to the stock put in.

Let $3x=$ A's share of the gain, then $4x=$ B's, and $5x=$ C's. (See Example 24, page 126.)

Then $3x+4x+5x=60$

or, $12x=60$

$x=5$

$3x=15$, A's share.

$4x=20$, B's "

$5x=25$, C's "

2. To make this question general, suppose A puts in m dollars, as often as B puts in n dollars, and as often as C puts in r dollars; and that they gain c dollars. To find the share of each.

REVIEW.—166. By what rule do you find two quantities, when their sum and difference are given? 167. When the times are given, in which each of two agents can produce a certain effect, how is the time found in which they can jointly produce it?

Let the share of A be denoted by mx, then $nx=$ B's, and $rx=$ C's share. Then $mx+nx+rx=c$

$$x=\frac{c}{m+n+r}$$

$$mx=m\times\frac{c}{m+n+r}=\frac{mc}{m+n+r}$$

$$nx=n\times\frac{c}{m+n+r}=\frac{nc}{m+n+r}$$

$$rx=r\times\frac{c}{m+n+r}=\frac{rc}{m+n+r}.$$

By examining these formula, we see that the whole gain, c, is divided by $m+n+r$, the sum of the proportions of stock furnished by all the partners, and that this quotient is multiplied by m, n, and r, each one's respective proportion, to obtain his share of the gain.

If c had represented loss, instead of gain, the same solution would have applied. Hence, to find each partner's share of the gain or loss, we have the following

RULE.

Divide the whole gain or loss by the sum of the proportions of stock, and multiply the quotient by each partner's proportion, to obtain his respective share.

When the times in which the respective stocks are employed are different, it becomes necessary to reduce them to the same time, to ascertain what proportion they bear to each other.

Thus, if A have 3 dollars in trade 4 months, and B 2 dollars 5 months, we see, that 3 dollars for 4 months, are the same as 12 dollars for 1 month; and 2 dollars for 5 months, are the same as 10 dollars for one month. Therefore, in this case, the gain or loss must be divided in the proportion of 12 to 10; that is, in proportion to the product of the stocks by the times in which they were employed. Hence, when time in fellowship is considered, we have the following

RULE.

Multiply each man's stock by the time during which it was employed; and then, according to the preceding rule, divide the gain or loss in proportion to these products.

3. A, B, and C engaged in trade; A put in 200 dollars, B 300, and C 700; they lost 60 dollars; what was each man's share?

Ans. A's loss \$10, B's \$15, and C's \$35.

REVIEW.—168. How is the gain or loss in fellowship found, when the times in which the stock is employed are the same? How is it found, when the times are different?

Since the sums engaged, evidently are to each other, as 2, 3, and 7, we may either use these numbers, or those representing the stock.

4. In a trading expedition A put in 200 dollars 3 months, B 150 dollars for 5 months, and C 100 dollars for 8 months; they gained 215 dollars; what was each man's share of the gain?

Ans. A's share \$60, B's \$75, and C's \$80.

Art. **169.**—1. Two men, A and B, can perform a certain piece of work in a days, A and C in b days, and B and C in c days; in what time could each one, alone, perform it? and, in what time could they perform it, all working together?

Let x, y, and z represent the days in which A, B, and C can respectively do it.

Then $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, represent the parts of the work which A, B, and C can each do in 1 day.

Since A and B can do it in a days, they do $\frac{1}{a}$ part of it in 1 day. But, $\frac{1}{x}+\frac{1}{y}$ represents the part of the work which A and B can do in one day. Hence,

$\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$ (1.) and reasoning in a similar manner, we have

$\frac{1}{x}+\frac{1}{z}=\frac{1}{b}$ (2.)

$\frac{1}{y}+\frac{1}{z}=\frac{1}{c}$ (3.)

$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, by adding the three equations together.

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$ (4.) by dividing by 2.

$\frac{1}{x}=\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2c}=\frac{bc+ac-ab}{2abc}$, by subtracting (3) from (4).

or, $x(ac+bc-ab)=2abc$, by clearing of fractions.

$x=\frac{2abc}{ac+bc-ab}$. In a similar manner, by subtracting equation (2) from (4), and reducing, we find $y=\frac{2abc}{ab+bc-ac}$.

Also, in the same manner, z is found $=\frac{2abc}{ab+ac-bc}$.

Since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$, or $\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$, represents the part all can do in

one day; if we divide 1 by $\left(\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}\right)$, the quotient, which is $\frac{2abc}{ab+ac+bc}$, will represent the number of days in which all can perform it.

ART. **170.**—In the solution of questions, it is sometimes necessary to use general values for particular quantities, to ascertain the relation which they bear to each other; as in the following problem.

If 4 acres pasture 40 sheep 4 weeks, and 8 acres pasture 56 sheep 10 weeks, how many sheep will 20 acres pasture 50 weeks, the grass growing uniformly all the time?

The chief difficulty in solving this question, consists in ascertaining the relation that exists between the original quantity of grass on an acre, and the growth on each acre in one week.

Let $m=$ the quantity on an acre when the pasturage began, and $n=$ the growth on 1 acre in 1 week; m and n representing pounds, or any other measure of the quantity of grass.

Then $4n=$ the growth on 1 acre in 4 weeks.

And $16n=$ the growth on 4 acres in 4 weeks.

Also, $4m+16n=$ the whole amount of grass on 4 acres in 4 weeks.

If 40 sheep eat $4m+16n$ in 4 weeks, then 40 sheep eat $\frac{4m+16n}{4}=m+4n$ in one week.

And 1 sheep eats $\frac{m+4n}{40}=\frac{m}{40}+\frac{n}{10}$ in one week.

Again, $8m+80n=$ the whole amount of grass on 8 acres in 10 weeks.

If 56 sheep eat $8m+80n$ in 10 weeks,

Then 56 sheep eat $\frac{8m}{10}+8n$ in 1 week.

And 1 sheep eats $\frac{8m}{560}+\frac{8n}{56}=\frac{m}{70}+\frac{n}{7}$ in 1 week.

Hence, $\frac{m}{40}+\frac{n}{10}=\frac{m}{70}+\frac{n}{7}$.

Or, $7m+28n=4m+40n$

$3m=12n$

$m=4n$

or $n=\frac{1}{4}m$; hence, the growth on one acre in 1 week, is equal to $\frac{1}{4}$ of the original quantity on an acre.

Then, 1 sheep, in 1 week, eats $\frac{m}{40}+\frac{n}{10}=\frac{m}{40}+\frac{m}{40}=\frac{m}{20}$.

And 1 sheep, in 50 weeks, eats $\frac{m}{20}\times50=\frac{5m}{2}$.

20 acres have an original quantity of grass, denoted by $20m$. The growth of 1 acre in 1 week being $\frac{1}{4}m$, in 50 weeks, it will be $\frac{50m}{4}$. And the growth of 20 acres, in 50 weeks, will be

$$\frac{50m}{4}\times20=250m.$$

Then $20m+250m=270m$, the whole amount of grass on 20 acres in 8 weeks.

Then $270m\div\frac{5m}{2}=\frac{540m}{5m}=108$, the number of sheep required

GENERAL PROBLEMS.

1. Divide the number a into two parts, so that one of them shall be n times the other. Ans. $\frac{na}{n+1}$ and $\frac{a}{n+1}$.

2. Divide the number a into two parts, so that m times one part shall be equal to n times the other. Ans. $\frac{na}{m+n}$ and $\frac{ma}{m+n}$.

3. Divide the number a into two parts, so that when the first is multiplied by m, and the second by n, the sum of the products may be equal to b. Ans. $\frac{b-na}{m-n}$ and $\frac{ma-b}{m-n}$.

4. Find a number, which being divided by m, and by n, the sum of the quotients shall be equal to a. Ans. $\frac{mna}{m+n}$.

5. Divide a into three such parts, that the second shall be m, and the third n times the first.

Ans. $\frac{a}{1+m+n}$, $\frac{ma}{1+m+n}$, and $\frac{na}{1+m+n}$.

6. Divide a into two such parts, that one of them being divided by b, and the other by c, the sum of the quotients shall be equal to d. Ans. $\frac{b(a-cd)}{b-c}$ and $\frac{c(bd-a)}{b-c}$.

7. What number must be added to a and b, so that the sums shall be to each other as m to n? Ans. $\frac{mb-na}{n-m}$.

8. What number must be subtracted from a and b, so that the differences shall be to each other as m to n? Ans. $\frac{na-mb}{n-m}$.

9. What number must be added to a, and subtracted from b, that the sum may be to the difference as m to n? Ans. $\frac{mb-na}{m+n}$.

10. After paying away $\frac{1}{m}$ and $\frac{1}{n}$ of my money, I had a dollars left; how many dollars had I at first? Ans. $\frac{mna}{mn-m-n}$.

11. What quantity is that of which the $\frac{m}{n}$ part, diminished by the $\frac{p}{q}$ part, is equal to a? Ans. $\frac{anq}{mq-np}$.

12. A certain number of persons paid for the use of a boat, for a pleasure excursion, a cents each; but, if there had been b persons less, each would have had to pay c cents; how many persons were there? Ans. $\frac{bc}{c-a}$.

13. A person gave some poor persons a cents a piece, and had b cents left; but, if he had given them c cents a piece, he would have had d cents left; how many persons were there? Ans. $\frac{d-b}{a-c}$

14. A farmer mixes oats at a cents per bushel, with rye at b cents per bushel, so that a bushel of the mixture is worth c cents; how many bushels of each will n bushels of the mixture contain? Ans. $\frac{n(c-b)}{a-b}$ and $\frac{n(a-c)}{a-b}$

15. A person borrowed as much money as he had in his purse, and then spent a cents; again, he borrowed as much as he had in his purse, after which he spent a cents; he borrowed and spent, in the same manner, a third and fourth time, after which, he had nothing left; how much had he at first? Ans. $\frac{15a}{16}$.

16. A person has 2 kinds of coin; it takes a pieces of the first, and b pieces of the second, to make one dollar; how many pieces of each kind must be taken, so that c pieces may be equivalent to a dollar? Ans. $\frac{a(b-c)}{b-a}$ and $\frac{b(c-a)}{b-a}$.

ART. **171.**—It sometimes happens in the solution of an equation of the first degree, that the second or some higher power of the unknown quantity occurs; but, in such a manner, that it is easily removed, or made to disappear, so that the equation can be solved in the usual manner. The following are examples of equations and problems belonging to this class.

1. Given $2x^2+8x=11x^2-10x$, to find the value of x.

By dividing each side by x, we have

$$2x+8=11x-10, \text{ from which } x=2.$$

2. Given $(4+x)(3+x)-6(10-x)=x(7+x)$, to find x.

Performing the operations indicated, we have

$$12+7x+x^2-60+6x=7x+x^2$$

Omitting the quantities on each side which are equal, we have

$$12-60+6x=0, \text{ from which } x=8.$$

3. $3x^2-8x=24x-5x^2$. . . . . Ans. $x=4$

4. $40x^2-6x^3-16x^2=120x^2-14x^3$. . . . . Ans. $x=12$

5. $3ax^3-10ax^2=8ax^2+ax^3$. . . . . Ans. $x=9$.

6. $x^2+\frac{2x^2}{3}-\frac{x^2}{2}=x$. . . . . Ans. $x=\frac{6}{7}$

7. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$. . . . . Ans. $x=20$.

8. $(a+x)(b+x)-a(c-b)=x(b+x)$. . . . . Ans. $x=c-2b$.

9. $ax+b^2=\frac{a(x^2+c^2)}{a+x}$. . . . . Ans. $x=\frac{a(c^2-b^2)}{a^2+b^2}$.

10. $x+a+b+c=\frac{x^2+a^2+b^2+c^2}{a+b-c+x}$. . . . . . . Ans. $x=\frac{c^2-ab}{a+b}$.

11. The difference between two numbers is 2, and their product is 8 greater than the square of the less; what are the numbers?

Ans. 4 and 6.

12. It is required to divide the number a into two such parts, that the difference of their squares may be c.

Ans. $\frac{a^2-c}{2a}$ and $\frac{a^2+c}{2a}$.

13. If a certain book contained 5 more pages, with 10 more lines on a page, the number of lines would be increased 450; but if it contained 10 pages less, with 5 lines less on a page, the whole number of lines would be diminished 450. Required the number of pages, and the number of lines on a page.

Ans. 20 pages, and 40 lines on a page.

NEGATIVE SOLUTIONS.

Art. **172.**—It has been stated already (Art. 23), that when a quantity has no sign prefixed, the sign *plus* is understood; and also (Art. 64), that all numbers or quantities are regarded as positive, unless they are otherwise designated. Hence, in all problems, it is understood, that the results are required in positive numbers. It sometimes happens, however, that the value of the unknown quantity in the solution of a problem, is found to be *minus*. Such a result is termed a *negative solution*. We shall now examine a question of this kind.

1. What number must be added to the number 5, that the sum shall be equal to 3?

Let $x=$ the number.

Then $5+x=3$.

And $x=3-5=-2$.

Now, -2 added to 5, according to the rule for Algebraic Addition, gives a sum equal to 3; thus, $5+(-2)=3$. The result, -2, is said to satisfy the question in an *algebraic sense;* but the problem is evidently impossible in an *arithmetical sense,* since any positive number added to 5, must *increase,* instead of *diminishing* it; and this impossibility is shown, by the result being negative, instead of positive. Since adding -2, is the same as subtracting $+2$ (Art. 61), the result is the answer to the following question: What number must be *subtracted* from 5, that the remainder may be equal to 3?

Let the question now be made general, thus:

What number must be added to the number a, that the sum shall be equal to b?

Let $x=$ the number.

Then $a+x=b$.

And $x=b-a$.

Now, since $a+(b-a)=b$, this value of x will always satisfy the question in an algebraic sense.

While b is greater than a, the value of x will be *positive,* and, whatever values are given to b and a, the question will be consistent, and can be answered in an *arithmetical sense.* Thus, if $b=10$, and $a=8$, then $x=2$.

But if b becomes less than a, the value of x will be *negative;* and whatever values are given to b and a, the result obtained, will satisfy the question in its *algebraic,* but not in its *arithmetical sense.*

Thus, if $b=5$, and $a=8$, then $x=-3$. Now $8+(-3)=5$; that is, if we *subtract* 3 from 8, the remainder is 5. We thus see, that when a becomes greater than b, the question, to be consistent, should read, What number must be *subtracted* from the number a, that the *remainder* shall be equal to b? From this we see,

1st. *That a negative solution indicates some inconsistency or absurdity, in the question from which the equation was derived.*

2d. *When a negative solution is obtained, the question, to which it is the answer, may be so modified as to be consistent.*

Let the pupil now read, carefully, the "OBSERVATIONS ON ADDITION AND SUBTRACTION," page 43, and then modify the following questions, so that they shall be consistent, and the results true in an *arithmetical sense.*

2. What number must be *subtracted* from 20, that the *remainder* shall be 25? ($x=-5$.)

REVIEW.—172. What is a negative solution? When is a result said to satisfy a question in an algebraic sense? In an arithmetical sense? What does a negative solution indicate?

3. What number must be *added* to 11, that the *sum* being multiplied by 5, the product shall be 40? ($x=-3$.)

4. What number is that, of which the $\frac{2}{3}$ *exceeds* the $\frac{3}{4}$ by 3? ($x=-36$.)

5. A father, whose age is 45 years, has a son, aged 15; *in how many years* will the son be $\frac{1}{4}$ as old as his father? ($x=-5$.)

DISCUSSION OF PROBLEMS.

ART. **173.**—When a question has been solved in a general manner, that is, by representing the known quantities by letters, we may inquire what values the results will have, when particular suppositions are made with regard to the known quantities. The determination of these values, and the examination of the various results which we obtain, constitute what is termed the *discussion* of the problem.

The various forms which the value of the unknown quantity may assume, are shown in the discussion of the following question.

1. After subtracting b from a, what number, multiplied by the remainder, will give a product equal to c?

Let $x=$ the number.

Then $(a-b)x=c$.

$$x=\frac{c}{a-b}.$$

Now, this result may have five different forms, depending on the values of a, b, and c.

NOTE.—In the following forms, A denotes merely some quantity.

1st. When b is less than a. This gives positive values, of the form $+A$.

2d. When b is greater than a. This gives negative values, of the form $-A$.

3d. When b is equal to a. This gives values of the form $\frac{A}{0}$.

4th. Where c is 0, and b either greater or less than a. This gives values of the form $\frac{0}{A}$.

5th. When b is equal to a, and c is equal to 0. This gives values of the form $\frac{0}{0}$.

We shall examine each of these in succession.

I. When b is less than a.

In this case, $a-b$ is positive, and the value of x is positive. To illustrate this form, let $a=8$, $b=3$, and $c=20$, then $x=4$.

REVIEW.—172. When a negative solution is obtained, how may the question, to which it is the answer, be modified? 173. What do you understand by the discussion of a problem? The expression c divided by $a-b$, may have how many forms? Name these different forms.

II. When b is greater than a.

In this case, $a-b$ is a negative quantity, and the value of x will be negative. This evidently should be so, since minus multiplied by minus produces plus; that is, if $a-b$ is *minus*, x must be *minus*, in order that their product shall be equal to c, a positive quantity. To illustrate this case by numbers, let $a=2$, $b=5$, and $c=12$; then, $a-b=-3$, $x=-4$, and $-3\times-4=12$.

III. When b is equal to a.

In this case x becomes equal to $\frac{c}{0}$. We must now inquire, what is the value of a fraction when the denominator is zero.

1st. Suppose the denominator 1, then $\frac{c}{1}=c$.

2d. Suppose the denominator $\frac{1}{10}$, then $\frac{c}{.1}=10c$.

3d. Suppose the denominator $\frac{1}{100}$, then $\frac{c}{.01}=100c$.

4th. Suppose the denominator $\frac{1}{1000}$, then $\frac{c}{.001}=1000c$.

While the numerator remains the same, we see, that as the denominator *decreases*, the value of the fraction *increases*. Hence, if the denominator be *less* than any assignable quantity, that is 0, the value of the fraction will be *greater* than any assignable quantity, that is, infinitely great. This is designated by the sign ∞, that is

$$\frac{c}{0}=\infty.$$

This is interpreted by saying, that no finite value of x will satisfy the equation; that is, there is no number, which being multiplied by 0, will give a product equal to c.

IV. When c is 0, and b is either greater or less than a.

If we put $a-b$ equal to d, then $x=\frac{0}{d}=0$, since $d\times0=0$; that is, when the product is zero, one of the factors must be zero.

V. When $b=a$, and $c=0$.

In this case, we have $x=\frac{c}{a-b}=\frac{0}{0}$, or $x\times0=0$.

Since any quantity multiplied by 0, gives a product equal to 0, *any finite value* of x whatever, will satisfy this equation; hence, x is indeterminate. On this account, we say that $\frac{0}{0}$ is the symbol of indetermination; that is, the quantity which it represents, has no particular value.

REVIEW.—173. When is x of the form $+$A? When is x of the form $-$A? When is x of the form $\frac{A}{0}$, or ∞? Show how the value of a fraction increases, as its denominator decreases. What is the value of a fraction whose denominator is zero? Of x when c is 0, and b greater or less than a?

The form $\frac{0}{0}$ sometimes arises from a particular supposition, when the terms of a fraction contain a common factor. Thus, if $x=\frac{a^2-b^2}{a-b}$, and we make $b=a$, it reduces to $\frac{a^2-a^2}{a-a}=\frac{0}{0}$; but, if we cancel the common factor, $a-b$, and then make $b=a$, we have $x=2a$. This shows, that before deciding the value of the unknown quantity to be indeterminate, we must see that this apparent indetermination has not arisen from the existence of a factor, which, by a particular supposition, becomes equal to zero.

The discussion of the following problem, which was originally proposed by Clairaut, will serve to illustrate further the preceding principles, and show, that the results of every correct solution, correspond to the circumstances of the problem.

PROBLEM OF THE COURIERS.

Two couriers depart at the same time, from two places, A and B, distant a miles from each other; the former travels m miles an hour, and the latter, n miles; where will they meet?

There are two cases of this question.

I. When the couriers travel toward each other.

Let P be the point where they meet, and a=AB, the distance between the two places.

Let x=AP, the distance which the first travels.

Then $a-x$=BP, the distance which the second travels.

Then, the distance each travels, divided by the number of miles traveled in an hour, will give the number of hours he was traveling.

Therefore, $\frac{x}{m}$= the number of hours the first travels.

And $\frac{a-x}{n}$= the number of hours the second travels.

But they both travel the same number of hours, therefore

$$\frac{x}{m}=\frac{a-x}{n}$$

$$nx=am-mx$$

$$x=\frac{am}{m+n}$$

$$a-x=\frac{an}{m+n}.$$

1st. Suppose $m=n$, then $x=\frac{am}{2m}=\frac{a}{2}$, and $a-x=\frac{a}{2}$; that is, if the couriers travel at the same rate, each travels precisely half the distance.

2d. Suppose $n=0$, then $x=\frac{am}{m}=a$; that is, if the second courier remains at rest, the first travels the whole distance from A to B. Both these results are evidently true, and correspond to the circumstances of the problem.

II. When the couriers travel in the same direction.

As before, let P be the point of meeting, each traveling in that direction, and let a=AB the distance between the places.

x=AP the distance the first travels.

$x-a$=BP the distance the second travels.

Then, reasoning as in the first case, we have

$$\frac{x}{m}=\frac{x-a}{n}$$

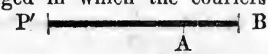
$$nx=mx-am$$

$$x=\frac{am}{m-n}, \text{ and } x-a=\frac{an}{m-n}.$$

1st. If we suppose m greater than n, the value of x will be positive; that is, the couriers will meet on the right of B. This evidently corresponds to the circumstances of the problem.

2d. If we suppose n greater than m, the value of x, and also that of $x-a$, will be negative. This negative value of x shows that there is some inconsistency in the question (Art. 172). Indeed, when m is less than n, it is evident that the couriers can not meet, since the forward courier is traveling faster than the hindmost. Let us now inquire how the question may be modified, so that the value obtained for x shall be consistent.

If we suppose the direction changed in which the couriers travel; that is, that the first travels from A, and the second from B toward P′; and that a=AB



x=AP′

$a+x$=BP′, we have, reasoning as before,

$$\frac{x}{m}=\frac{a+x}{n}$$

$$x=\frac{am}{n-m}, \text{ and } a+x=\frac{an}{n-m}.$$

The distances traveled are now both positive, and the question will be consistent, if we regard the couriers, instead of traveling toward P, as traveling in the opposite direction toward P′. The change of sign, thus indicating a change of direction (Art. 64).

3d. If we suppose m equal to n.

In this case x is equal to $\frac{am}{0}$, and $x-a=\frac{an}{0}$.

As has been already shown (Art. 173), when the unknown quantity takes this form, it is not satisfied by any finite value; or, it is infinitely great. This evidently corresponds to the circumstances of the problem; for, if the couriers travel at the same rate, the one can *never* overtake the other. This is sometimes otherwise expressed, by saying, they only meet at an *infinite* distance from the point of starting.

4th. If we suppose $a=0$, then $x=\frac{0}{m-n}$, and $x-a=\frac{0}{m-n}$.

When the unknown quantity takes this form, it has been shown already, that its value is 0. This corresponds to the circumstances of the problem; for, if the couriers are *no* distance apart, they will have to travel *no* (0) distance to be together.

5th. If we suppose $m=n$, and $a=0$.

In this case, $x=\frac{0}{0}$, and $x-a=\frac{0}{0}$. When the unknown quantity takes this form, it has been shown (Art. 173), that it may have *any finite value* whatever. This, also, evidently corresponds to the circumstances of the problem; for, if the couriers are *no* distance apart, and travel at the *same* rate, they will be *always* together; that is, at *any* distance whatever from the point of starting.

Lastly, if we suppose $n=0$, then $x=\frac{am}{m}=a$; that is, the first courier travels from A to B, overtaking the second at B.

If we suppose $n=\frac{m}{2}$, then $x=\frac{2am}{m}=2a$, and the first travels twice the distance from A to B, before overtaking the second. Both results evidently correspond to the circumstances of the problem.

CASES OF INDETERMINATION IN EQUATIONS OF THE FIRST DEGREE, AND IMPOSSIBLE PROBLEMS.

Art. 174.—An equation is termed *independent*, when the relation of the quantities which it contains, can not be obtained directly from others with which it is compared. Thus, the equation

$$x+2y=11$$
$$2x+5y=26$$

are independent of each other, since the one can not be obtained from the other in a direct manner.

REVIEW.—173. What is the value of x when $b=a$ and $c=0$? What is the value of a fraction whose terms are both zero? Show, that this form sometimes arises from the existence of a common factor, which, by a particular hypothesis, reduces to zero. Discuss the problem of the "Couriers," and show, that in every hypothesis the solution corresponds to the circumstances of the problem.

The equations, $x+2y=11$

$2x+4y=22$, are not independent of each other, the second being derived directly from the first, by multiplying both sides by 2.

ART. **175.**—An equation is said to be *indeterminate*, when it can be verified by different values of the same unknown quantity. Thus, in the equation $x-y=5$, by transposing y, we have $x=5+y$.

If we make $y=1$, $x=6$. If we make $y=2$, $x=7$, and so on; from which it is evident, that an *unlimited* number of values may be given to x and y, that will verify the equation.

If we have two equations containing three unknown quantities, we may eliminate one of them; this will leave a single equation, containing two unknown quantities, which, as in the preceding example, will be indeterminate.

Thus, if we have $x+3y+z=10$

and $x+2y-z=\ 6$, if we eliminate x we have

$y+2z=\ 4$, from which $y=4-2z$.

If we make $z=1$, $y=2$, and $x=10-3y-z=3$.

If we make $z=1\frac{1}{2}$, $y=1$, and $x=5\frac{1}{2}$.

In the same manner, an unlimited number of values of the three unknown quantities may be found, that will verify both equations. Other examples might be given, but these are sufficient to show, that *when the number of unknown quantities exceeds the number of independent equations, the problem is indeterminate.*

A question is sometimes indeterminate that involves only one unknown quantity; the equation deduced from the conditions, being of that class denominated identical. The following is an example.

What number is that, of which the $\frac{3}{4}$, diminished by the $\frac{2}{3}$, is equal to the $\frac{1}{20}$ increased by the $\frac{1}{30}$?

Let $x=$ the number.

Then $\frac{3x}{4}-\frac{2x}{3}=\frac{x}{20}+\frac{x}{30}$.

Clearing of fractions, $45x-40x=3x+2x$

or, $5x=5x$, which will be verified by any value of x whatever.

ART. **176.**—The reverse of the preceding case requires to be considered; that is, when the number of equations is greater than the number of unknown quantities. Thus, we may have

$$x+\ y=10 \quad (1.)$$
$$x-\ y=\ 4 \quad (2.)$$
$$2x-3y=\ 5 \quad (3.)$$

Each of these equations being independent of the other two, one of them is unnecessary, since the values of x and y, which are 7 and 3, may be determined from any two of them. When a

problem contains more conditions than are necessary for determining the values of the unknown quantities, those that are unnecessary, are termed *redundant conditions.*

The number of equations may exceed the number of unknown quantities, so that the values of the unknown quantities shall be incompatible with each other. Thus, if we have

$$x+\ y=\ 9\quad (1.)$$
$$x+2y=13\quad (2.)$$
$$2x+3y=21\quad (3.)$$

The values of x and y, found from equations (1) and (2), are $x=5$, $y=4$; from equations (1) and (3), are $x=6$, $y=3$; and from equations (2) and (3), are $x=3$, $y=5$. From this it is manifest, that only two of these equations can be true at the same time.

A question that contains only one unknown quantity, is sometimes impossible. The following is an example.

What number is that, of which the $\frac{1}{2}$ and $\frac{1}{3}$ diminished by 4, is equal to the $\frac{5}{6}$ increased by 8?

Let $x=$ the number, then $\frac{x}{2}+\frac{x}{3}-4=\frac{5x}{6}+8$.

Clearing of fractions, $3x+2x-24=5x+48$.

by subtracting equals from each side, $0=72$; which shows, that the question is absurd.

REMARK.—Problems from which contradictory equations are deduced, are termed *irrational* or *impossible.* The pupil should be able to detect the character of such questions when they occur, in order that his efforts may not be wasted, in an attempt to perform an impossibility. A careful study of the preceding principles, will enable him to do this, so far as equations of the first degree are concerned.

ART. **177.**—Take the equation $ax-cx=b-d$, in which a represents the sum of the positive, and $-c$ the sum of the negative coëfficients of x; b the sum of the positive, and $-d$ the sum of the negative known quantities. This will evidently express a simple equation involving one unknown quantity, in its most general form.

This gives $(a-c)x=b-d$.

Let $a-c=m$, and $b-d=n$, we then have $mx=n$, or $x=\frac{n}{m}$.

Now, since n divided by m can give but one quotient, we infer that *an equation of the first degree has but one root;* that is, in a simple equation involving but one unknown quantity, there is but one value that will verify the equation.

REVIEW.—174. When is an equation termed independent? Give an example. 175. When is an equation said to be indeterminate? Give an example. 176. What are redundant conditions?

CHAPTER VI.

FORMATION OF POWERS—

EXTRACTION OF THE SQUARE ROOT—RADICALS OF THE SECOND DEGREE.

INVOLUTION, OR FORMATION OF POWERS.

ART. **178.**—The term *power* is used to denote the product arising from multiplying a quantity by itself, a certain number of times; and the quantity which is multiplied by itself, is called the *root* of the power.

Thus a^2 is called the *second* power of a, because a is taken *twice* as a factor; and a is called the *second* root of a^2.

So, also, a^3 is called the *third* power of a, because $a\times a\times a=a^3$, the quantity a being taken *three* times as a factor; and a is called the *third* root of a^3.

The second power is generally called the *square*, and the second root, the *square* root. In like manner, the third power is called the *cube*, and the third root, the *cube* root.

The figure indicating the power to which the quantity is to be raised, is called the *index*, or *exponent;* it is to be written on the right, and a little higher than the quantity. (See Articles 33 and 35.)

REMARK.—A power may be otherwise defined thus: *The* n*th power of a quantity, is the product of* n *factors, each equal to the quantity;* where n may be any number, as 2, 3, 4, and so on. Therefore, *we may obtain any power of a quantity by taking it as a factor as many times as there are units in the exponent of the power to which it is to be raised.* This rule alone, is sufficient for every question in the formation of powers; but, for the more easy comprehension of pupils, it is generally presented in detail, as in the following cases.

CASE I.

TO RAISE A MONOMIAL TO ANY GIVEN POWER.

ART. **179.**—1. Let it be required to raise $2ab^2$ to the third power.

According to the definition, the third power of $2ab^2$, will be the product arising from taking it *three* times as a factor. Thus,

$$(2ab^2)^3=2ab^2\times2ab^2\times2ab^2=2\times2\times2aaab^2b^2b^2$$
$$=2^3\times a^{1+1+1}\times b^{2+2+2}=2^3\times a^{1\times3}\times b^{2\times3}=8a^3b^6.$$

In this example, we see, that the coëfficient of the power is found

by raising the coëfficient, 2, of the root, to the given power; and, that the exponent of each letter is obtained, by multiplying the exponent of the letter in the root, by 3, the index of the required power.

ART. **180.**—With regard to the signs of the different powers, there are two cases.

First, when the root is *positive;* and second, when the root is *negative.*

1st. When the root is positive. Since the product of any number of positive factors is always positive, it is evident, that if the root is positive, all the powers will be positive.

Thus, $+a\times+a=+a^2$

$+a\times+a\times+a=+a^3$, and so on.

2d. When the root is negative. Let us examine the different powers of a negative quantity, as $-a$.

$-a=$ first power, *negative.*

$-a\times-a=+a^2=$ second power, *positive.*

$-a\times-a\times-a=-a^3=$ third power, *negative.*

$-a\times-a\times-a\times-a=+a^4=$ fourth power, *positive.*

$-a\times-a\times-a\times-a\times-a=-a^5=$ fifth power, *negative.*

From this, we see, that the product of an *even* number of negative factors is *positive,* and that the product of an *odd* number of negative factors is *negative.* Therefore, the even powers of a negative quantity are all *positive,* and the odd powers are all *negative.* Hence we have the following

RULE,

FOR RAISING A MONOMIAL TO ANY GIVEN POWER.

Raise the numeral coëfficient to the required power, and multiply the exponent of each of the letters, by the exponent of the power. If the monomial is positive, all the powers will be positive; but, if it is negative, all the even powers will be positive, and all the odd powers negative.

EXAMPLES.

1. Find the square of $3ax^2y^3$. . . . . Ans. $9a^2x^4y^6$.
2. Find the square of $5b^2c^3$. . . . . Ans. $25b^4c^6$.
3. Find the cube of $2x^2y^3$. . . . . Ans. $8x^6y^9$
4. Find the square of $-ab^2c$. . . . . Ans. $a^2b^4c^2$
5. Find the cube of $-abc^2$. . . . . Ans. $-a^3b^3c^6$

REVIEW.—177. Show, that in an equation of the first degree, the unknown quantity can have but one value. 178. What does the term power denote? The term root? What is the second power of a? Why? The third power of a? Why? What is the second power generally called? The second root? What is the index or exponent? Where should it be written?

6. Find the fourth power of $3ab^3c^2$. . . . . Ans. $81a^4b^{12}c^8$.
7. Find the fourth power of $-3ab^3c^2$. . . . . Ans. $81a^4b^{12}c^8$.
8. Find the fifth power of ab^3cd^2. . . . . . Ans. $a^5b^{15}c^5d^{10}$.
9. Find the fifth power of $-ab^3cd^2$. . . . Ans. $-a^5b^{15}c^5d^{10}$.
10. Find the sixth power of a^2bc^3d. . . . . . Ans. $a^{12}b^6c^{18}d^6$.
11. Find the seventh power of $-m^2n^3$. . . . . Ans. $-m^{14}n^{21}$.
12. Find the eighth power of $-mn^2$. . . . . Ans. m^8n^{16}.
13. Find the cube of $-3a^4$. . . . . Ans. $-27a^{12}$.
14. Find the cube of $-3xy^2$. . . . . Ans. $-27x^3y^6$.
15. Find the fourth power of $5a^2x^3$. . . . . . . Ans. $625a^8x^{12}$.
16. Find the cube of $-4a^3x$. . . . . Ans. $-64a^9x^3$.
17. Find the cube of $-8x^3y^2$. . . . . Ans. $-512x^9y^6$.
18. Find the seventh power of $-2xyz^2$. . . Ans. $-128x^7y^7z^{14}$.
19. Find the fourth power of $7a^2x^3$. . . . . . . Ans. $2401a^8x^{12}$.
20. Find the fifth power of $-3a^2xy^2z^3$. . Ans. $-243a^{10}x^5y^{10}z^{15}$.

ART. 181. CASE II.

TO RAISE A POLYNOMIAL TO ANY POWER.

RULE.

Find the product of the quantity, taken as a factor as many times as there are units in the exponent of the power.

NOTE.—This rule, and that in the succeeding article, follow directly from the definition of a power.

EXAMPLES.

1. Find the square of $ax+cy$.

$(ax+cy)(ax+cy)=a^2x^2+2acxy+c^2y^2$. Ans.

2. Find the square of $1-x$. . . . . Ans. $1-2x+x^2$.
3. Find the square of $x+1$. . . . . Ans. x^2+2x+1.
4. Find the square of $ax-cy$. . . . . Ans. $a^2x^2-2acxy+c^2y^2$.
5. Find the square of $2x^2-3y^2$. . . . Ans. $4x^4-12x^2y^2+9y^4$.
6. Find the cube of $a+x$. . . . . Ans. $a^3+3a^2x+3ax^2+x^3$.
7. Find the cube of $x-y$. . . . . . . Ans. $x^3-3x^2y+3xy^2-y^3$.
8. Find the cube of $2x-1$. . . . . Ans. $8x^3-12x^2+6x-1$.
9. Find the fourth power of $c-x$.

Ans. $c^4-4c^3x+6c^2x^2-4cx^3+x^4$.

10. Find the square of $a+b+c$.

Ans. $a^2+2ab+b^2+2ac+2bc+c^2$.

11. Find the square of $a-b+c-d$.

Ans. $a^2-2ab+b^2+2ac-2ad+c^2-2bc+2bd-2cd+d^2$

12. Find the cube of $2x^2-3x+1$.

Ans. $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$.

ART. 182. CASE III.

TO RAISE A FRACTION TO ANY POWER.

RULE.

Raise both numerator and denominator to the required power by actual multiplication.

EXAMPLES.

1. Find the square of $\frac{a+b}{c-d}$.

$$\frac{a+b}{c-d}\times\frac{a+b}{c-d}=\frac{a^2+2ab+b^2}{c^2-2cd+d^2}.$$

2. Find the square of $\frac{2x}{3y}$. . . . . Ans. $\frac{4x^2}{9y^2}$.

3. Find the cube of $-\frac{ac}{x^2y}$. . . . . Ans. $-\frac{a^3c^3}{x^6y^3}$.

4. Find the square of $-\frac{2x^2}{3y}$. . . . . Ans. $\frac{4x^4}{9y^2}$.

5. Find the cube of $-\frac{4ay^3}{5z^2}$. . . . . Ans. $-\frac{64a^3y^9}{125z^6}$.

6. Find the square of $\frac{x-2}{x+3}$. . . . . Ans. $\frac{x^2-4x+4}{x^2+6x+9}$.

7. Find the cube of $\frac{2a(x-y)}{3yz^2}$. . Ans. $\frac{8a^3(x^3-3x^2y+3xy^2-y^3)}{27y^3z^6}$.

8. Find the square of $\frac{2(m-n)}{3(m+n)}$. . . . Ans. $\frac{4(m^2-2mn+n^2)}{9(m^2+2mn+n^2)}$.

BINOMIAL THEOREM.

ART. 183.—The Binomial Theorem (discovered by Sir Isaac Newton), explains the method of raising the sum or difference of any two quantities to any given power, by means of certain relations, that are always found to exist between the exponent of the power and the different parts of the required result.

To discover what these relations are, we shall first, by means of multiplication, find the different powers of a binomial, when both terms are positive; and next, when one term is positive, and the other negative.

REVIEW.—179. In raising $2ab^2$ to the third power, how is the coefficient of the power found? How is the exponent of each letter found? 180. When the root is positive, what is the sign of the different powers? When it is negative? What is the rule for raising a monomial to any given power? 181. What is the rule for raising a polynomial to any given power? 182. What is the rule for raising a fraction to any power? 183. What does the Binomial Theorem explain?

1. We will first raise $a+b$ to the fifth power.

$$
\begin{array}{l}
a+b \\
\underline{a+b} \\
a^2+ab \\
\underline{\quad +ab+b^2} \\
a^2+2ab+b^2= \text{. . . . . . . second power of } a+b\text{, or } (a+b)^2 \\
\underline{a+b} \\
a^3+2a^2b+ab^2 \\
\underline{\quad a^2b+2ab^2+b^3} \\
a^3+3a^2b+3ab^2+b^3= \text{. . . . third power of } a+b\text{, or } (a+b)^3. \\
\underline{a+b} \\
a^4+3a^3b+3a^2b^2+ab^3 \\
\underline{\quad +a^3b+3a^2b^2+3ab^3+b^4} \\
a^4+4a^3b+6a^2b^2+4ab^3+b^4= \text{. . . . . } (a+b)^4 \\
\underline{a+b} \\
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\underline{\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5} \\
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5= \text{. . . . . } (a+b)^5.
\end{array}
$$

The first letter, as a, is called the *leading* quantity; and the second letter, as b, the *following* quantity.

We will next raise $a-b$ to the fifth power.

$$
\begin{array}{l}
a-b \\
\underline{a-b} \\
a^2-ab \\
\underline{\quad -ab+b^2} \\
a^2-2ab+b^2= \text{. . . . . } (a-b)^2. \\
\underline{a-b} \\
a^3-2a^2b+ab^2 \\
\underline{\quad -a^2b+2ab^2-b^3} \\
a^3-3a^2b+3ab^2-b^3= \text{. . . . . } (a-b)^3. \\
\underline{a-b} \\
a^4-3a^3b+3a^2b^2-ab^3 \\
\underline{\quad -a^3b+3a^2b^2-3ab^3+b^4} \\
a^4-4a^3b+6a^2b^2-4ab^3+b^4= \text{. . . . . } (a-b)^4 \\
\underline{a-b} \\
a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\
\underline{\quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5} \\
a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5= \text{. . . . . } (a-b)^5.
\end{array}
$$

ART. **184.**—In examining the different parts of which these results consist, there are evidently *four* things to be considered.

1st. The *number* of terms of the power.

2d. The *signs* of the terms.

3d. The *exponents* of the letters.

4th. The *coëfficients* of the terms.

We shall examine these separately.

1st. Of the number of terms.

By examining either of these examples, we see, that the *second* power has *three* terms, the *third* power has *four* terms, the *fourth* power has *five* terms, the *fifth* power has *six* terms; hence, we infer, that *the number of terms in any power of a binomial, is one greater than the exponent of the power.*

2d. Of the signs of the terms.

From an examination of the examples, it is evident, that *when both terms of the binomial are positive, all the terms will be positive. When the first term is positive, and the second negative, all the* ODD *terms will be* POSITIVE, *and the* EVEN *terms* NEGATIVE.

NOTE.—By the *odd* terms are meant the 1st, 3d, 5th, and so on; and, by the *even* terms, the 2d, 4th, 6th, and so on.

3d. Of the exponents of the letters.

If we omit the coëfficients, the remaining parts of the fifth powers of $a+b$ and $a-b$, are

$(a+b)^5$ . . . . $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.

$(a-b)^5$ . . . . $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.

An examination of these and the other different powers of $a+b$ and $a-b$, shows, that the exponents of the letters are governed by the following laws:

1st. *The exponent of the leading letter in the first term, is the same as that of the power of the binomial; and the exponents of this letter in the other terms, decrease by unity from left to right, until the last term, which does not contain the leading letter.*

2d. *The exponent of the second letter in the second term is one; and the other exponents of this letter increase, by unity, from left to right, until the last term, in which the exponent is the same as that of the power of the binomial.*

3d. *The sum of the exponents of the two letters in any term is always the same, and is equal to the power of the binomial.*

REVIEW.—184. In examining the different powers of a binomial, what four things are to be considered? What is the number of terms in any power of a binomial? Give examples. When both terms of a binomial are positive, what are the signs of the terms? When one term is positive, and the other negative, what are the signs of the odd terms? Of the even terms? What is the exponent of the leading letter in the first term?

The pupil may now employ these principles, in writing the different powers of binomials without the coëfficients, as in the following examples.

$(x+y)^3$ . . . $x^3+x^2y+xy^2+y^3$.
$(x-y)^4$ . . . $x^4-x^3y+x^2y^2-xy^3+y^4$.
$(x+y)^5$ . . . $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$.
$(x-y)^6$ . . . $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.
$(x-y)^7$ . . . $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.
$(x+y)^8$ . . . $x^8+x^7y+x^6y^2+x^5y^3+x^4y^4+x^3y^5+x^2y^6+xy^7+y^8$.

4th. Of the coëfficients.

An inspection of the different powers of $(a+b)$ and $(a-b)$, plainly shows,

That the coëfficient of the first term is always 1; *and the coëfficient of the second term is the same as that of the power of the binomial.*

The law of the succeeding coëfficients is not so readily seen; it is, however, as follows:

If the coëfficient of any term be multiplied by the exponent of the leading letter, and the product be divided by the number of that term from the left, the quotient will be the coëfficient of the next term.

Omitting the coëfficients, the terms of $a+b$ raised to the sixth power, are $a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6$.

The coëfficients, according to the above principles, are

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6}.$$

$$\text{or, } 1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1.$$

Hence, $(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

From this, we see, that the coëfficients of the following terms are equal: the first and the last; the second from the first, and the second from the last; the third from the first and the third from the last, and so on. Hence, it is only necessary to find the coëfficients of half the terms, when their number is even, or one more than half, when their number is odd; the remaining coëfficients being equal to those already found.

EXAMPLES.

1. Raise $x+y$ to the third power. Ans. $x^3+3x^2y+3xy^2+y^3$.
2. Raise $(x-y)$ to the fourth power.
   Ans. $x^4-4x^3y+6x^2y^2-4xy^3+y^4$
3. Raise $m+n$ to the fifth power.
   Ans. $m^5+5m^4n+10m^3n^2+10m^2n^3+5mn^4+n^5$.

REVIEW.—184. How do the exponents of the leading letter decrease from left to right? What is the exponent of the second letter in the first term? In the second term? How do the exponents of the second letter increase from left to right? To what is the coëfficient of the first term equal?

4. Raise $x-z$ to the sixth power.

Ans. $x^6-6x^5z+15x^4z^2-20x^3z^3+15x^2z^4-6xz^5+z^6$.

5. What is the seventh power of $a+b$?

Ans. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

6. What is the eighth power of $m-n$? Ans. $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8$.

7. Find the ninth power of $x-y$. Ans. $x^9-9x^8y+36x^7y^2-84x^6y^3+126x^5y^4-126x^4y^5+84x^3y^6-36x^2y^7+9xy^8-y^9$.

8. Find the tenth power of $a+b$.

Ans. $a^{10}+10a^9b+45a^8b^2+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45a^2b^8+10ab^9+b^{10}$.

Art. **185.**—The Binomial Theorem may be used to find the different powers of a binomial, when one or both terms consist of two or more quantities.

1. Find the cube of $2x-ac^2$.

Let $2x=m$, and $ac^2=n$; then $2x-ac^2=m-n$.

$$(m-n)^3=m^3-3m^2n+3mn^2-n^3$$

$$\begin{array}{ll} m=2x & n=a\,c^2 \\ m^2=4x^2 & n^2=a^2c^4 \\ m^3=8x^3 & n^3=a^3c^6 \end{array}$$

Substituting these values of the different powers of m and n, in the equation above, and we have

$$(2x-ac^2)^3=8x^3-3\times4x^2\times ac^2+3\times2x\times a^2c^4-a^3c^6$$
$$=8x^3-12ac^2x^2+6a^2c^4x-a^3c^6.$$

2. Find the cube of $2a-3b$. Ans. $8a^3-36a^2b+54ab^2-27b^3$.

3. Find the fourth power of $m+2n$.

Ans. $m^4+8m^3n+24m^2n^2+32mn^3+16n^4$.

4. Find the third power of $4ax^2+3cy$.

Ans. $64a^3x^6+144a^2cx^4y+108ac^2x^2y^2+27c^3y^3$.

5. Find the fourth power of $2x-5z$.

Ans. $16x^4-160x^3z+600x^2z^2-1000xz^3+625z^4$.

Art. **186.**—The Binomial Theorem may likewise be used to raise a trinomial or quadrinomial to any power, by reducing it to a binomial by substitution, and then, after this has been raised to the required power, restoring the values of the letters.

1. Find the second power of $a+b+c$.

Let $b+c=x$; then $a+b+c=a+x$.

$$(a+x)^2=a^2+2ax+x^2$$
$$2ax=2a(b+c)$$
$$x^2=(b+c)^2=b^2+2bc+c^2$$

Then $(a+b+c)^2=a^2+2ab+2ac+b^2+2bc+c^2$.

Review.—184. Of the second term? How is the coëfficient of any other term found? Of what terms are the coëfficients equal?

2. Find the third power of $x+y+z$.

Ans. $x^3+3x^2y+3x^2z+3xy^2+6xyz+3xz^2+y^3+3y^2z+3yz^2+z^3$

3. Find the second power of $a+b+c+d$.

Ans. $a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2$

EXTRACTION OF THE SQUARE ROOT.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

Art. **187.**—The *second root*, or *square root* of a number, is that number, which being multiplied by *itself*, will produce the given number. Thus, 2 is the square root of 4, because $2\times2=4$.

The process of finding the second root of a given number, is called *the extraction of the square root.*

Art. **188.**—The first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

and their squares are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

The numbers in the first line, are also the square roots of the numbers in the second.

We see, from this, that the square root of a number between 1 and 4, is a number between 1 and 2; the square root of a number between 4 and 9, is a number between 2 and 3; the square root of a number between 9 and 16, is a number between 3 and 4, and so on.

Since the square root of 1 is 1, and of any number less than 100, is either one figure, or one figure and a fraction, therefore, *when the number of places of figures in a number is not more than* TWO, *the number of places of figures in the square root will be* ONE.

Again, take the numbers

10, 20, 30, 40, 50, 60, 70, 80, 90, 100,

their squares are

100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100, 10000.

From this we see, that the square root of 100 is ten; and of any number greater than 100, and less than 10000, the square root will be less than 100; that is, *when the number of places of figures is more than* TWO, *and not more than* FOUR, *the number of places of figures in the square root will be* TWO.

In the same manner, it may be shown, that when the number of places of figures in a given number are more than *four*, and not more than *six*, the number of places in the square root will be *three*, and so on. Or thus: when the number of places of figures

in the number is either *one* or *two*, there will be *one* figure in the root; when the number of places is either *three* or *four*, there will be *two* figures in the root; when the number of places is either *five* or *six*, there will be *three* figures in the root, and so on.

ART. **189.**—Every number may be regarded as being composed of tens and units. Thus, 23 consists of 2 tens and 3 units; 256 consists of 25 tens and 6 units. Therefore, if we represent the tens by t, and the units by u, any number will be represented by $t+u$, and its square, by the square of $t+u$, or $(t+u)^2$.

$$(t+u)^2=t^2+2tu+u^2=t^2+(2t+u)u.$$

Hence, *the square of any number is composed of the square of the tens, plus a quantity, consisting of twice the tens plus the units, multiplied by the units.*

Thus, the square of 23, which is equal to 2 tens and 3 units, is

$$\begin{array}{r} \text{2 tens squared} =(20)^2=400 \\ \text{(2 tens + 3 units) multiplied by } 3=(40+3)\times 3=\underline{129} \\ 529 \end{array}$$

1. Let it now be required to extract the square root of 529.

Since the number consists of three places of figures, its root will consist of two places, according to the principles in Art. 188; we therefore separate it into two periods, as in the margin.

$$\begin{array}{r|l} \dot{5}2\dot{9} & 23 \\ 400 & \\ \hline 20\times 2=40 & 129 \\ 3 & \\ \hline 43 & 129 \end{array}$$

Since the square of 2 tens is 400, and of 3 tens, 900, it is evident, that the greatest square contained in 500, is the square of 2 tens (20); the square of two tens (20) is 400; subtracting this from 529, the remainder is 129.

Now, according to the preceding theorem, this number 129 consists of twice the tens plus the units, multiplied by the units; that is, by the formula, it is $(2t+u)u$. Now, the product of the tens by the units can not give a product less than tens; therefore, the unit's figure (9) forms no part of the double product of the tens by the units. Then, if we divide the remaining figures (12) by the double of the tens, the quotient will be the unit's figure, or a figure greater than it.

REVIEW.—187. What is the square root of a number? Give an example. 188. When a number consists of only one figure, what is the greatest number of figures in its square? Give examples. When a number consists of two places of figures, what is the greatest number of figures in its square? Give examples. What relation exists between the number of places of figures in any number, and the number of places in its square? 189. Of what may every number be regarded as being composed? Prove this, and then illustrate it.

We then double the tens, which makes 4 ($2t$), and dividing this into 12, get 3 (u) for a quotient; this is the unit's figure of the root. This unit's figure (3) is to be added to the double of the tens (40), and the sum multiplied by the unit's figure. The double of the tens plus the units, is $40+3=43$ $(2t+u)$; multiplying this by 3 (u), the product is 129, which is the double of the tens plus the units, multiplied by the units. As there is nothing left after subtracting this from the first remainder, we conclude that 23 is the exact square root of 529.

In squaring the tens, and also in doubling them, it is customary to omit the ciphers, though they are understood. Also, the unit's figure is added to the double of the tens, by merely writing it in the unit's place. The actual operation is usually performed as in the margin.

```
    529|23
    4
 43|129
   |129
```

2. Let it be required to extract the square root of 55225.

Since this number consists of five places of figures, its root will consist of three places, according to the principles in Art. 188; we therefore separate it into *three* periods.

In performing this operation, we find the square root of the number 552, on the same principle as in the preceding example. We next consider the 23 as so many tens, and proceed to find the unit's figure (5) in the same manner as in the preceding example. Hence the

```
    55225|235
    4
 43|152
    129
465|2325
    2325
```

RULE,

FOR THE EXTRACTION OF THE SQUARE ROOT OF WHOLE NUMBERS.

1st. *Separate the given number into periods of two places each, beginning at the unit's place.* (The left period will often contain but one figure.)

2d. *Find the greatest square in the left period, and place its root on the right, after the manner of a quotient in division. Subtract the square of the root from the left period, and to the remainder bring down the next period for a dividend.*

3d. *Double the root already found, and place it on the left for a divisor. Find how many times the divisor is contained in the dividend, exclusive of the right hand figure, and place the figure in the root, and also on the right of the divisor.*

4th. *Multiply the divisor thus increased, by the last figure of the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

5th. *Double the whole root already found, for a new divisor, and continue the operation as before, until all the periods are brought down.*

REVIEW.—189. Extract the square root of 529, and show the reason for each step, by referring to the formula.

NOTE.—If, in any case, the dividend will not contain the divisor, the right hand figure of the former being omitted, place a zero in the root, and also at the right of the divisor, and bring down the next period.

ART. **190.**—In Division, when the remainder is greater than the divisor, the last quotient figure may be increased by at least 1; but in extracting the square root, the remainder may sometimes be greater than the last divisor, while the last figure of the root can not be increased. To know when any figure may be increased, the pupil must be acquainted with the relation that exists between the squares of two consecutive numbers.

Let a and $a+1$ be two consecutive numbers.

Then $(a+1)^2=a^2+2a+1$, is the square of the greater.

$(a)^2=a^2$ is the square of the less.

Their difference is $2a+1$.

Hence, *the difference of the squares of two consecutive numbers, is equal to twice the less number, increased by unity.* Consequently, when the remainder is less than twice the part of the root already found, plus unity, the last figure can not be increased.

Extract the square root of the following numbers.

1. 4225.	Ans. 65.	7. 678976. . .	Ans. 824.
2. 9409.	Ans. 97.	8. 950625. . . .	Ans. 975.
3. 15129. . . .	Ans. 123.	9. 363609. . . .	Ans. 603.
4. 120409. . .	Ans. 347.	10. 1525225. . .	Ans. 1235.
5. 289444. . .	Ans. 538.	11. 1209996225.	A. 34785.
6. 498436. . . .	Ans. 706.	12. 412252416.	Ans. 20304.

EXTRACTION OF THE SQUARE ROOT OF FRACTIONS.

ART. **191.**—Since $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$, therefore, the square root of $\frac{4}{9}$ is $\frac{2}{3}$, that is, $\sqrt{\frac{4}{9}}=\frac{\sqrt{4}}{\sqrt{9}}=\frac{2}{3}$. Hence, *when both terms of a fraction are perfect squares, its square root will be found, by extracting the square root of both terms.*

Before extracting the square root of a fraction, it should be reduced to its lowest terms, unless both numerator and denominator are perfect squares. The reason for this, will be seen by the following example.

Find the square root of $\frac{12}{27}$.

Here, $\frac{12}{27}=\frac{4\times3}{9\times3}$. Now, neither 12 nor 27 are perfect squares;

REVIEW.—189. What is the rule for extracting the square root of numbers? 190. What is the difference between the squares of two consecutive numbers? When may any figure of the quotient be increased?

but, by canceling the common factor 3, the fraction becomes $\frac{4}{9}$, of which the square root is $\frac{2}{3}$.

When both terms are perfect squares, and contain a common factor, the reduction may be made either before, or after the square root is extracted. Thus, $\sqrt{\frac{16}{36}}=\frac{4}{6}=\frac{2}{3}$; or, $\frac{16}{36}=\frac{4}{9}$, and $\sqrt{\frac{4}{9}}=\frac{2}{3}$.

Find the square root of each of the following fractions.

1. $\frac{81}{625}$. . . . . . Ans. $\frac{9}{25}$.
2. $\frac{64}{841}$. . . . . . Ans. $\frac{8}{29}$.
3. $\frac{1071}{2975}$. . . . . . Ans. $\frac{3}{5}$.
4. $\frac{572}{7007}$. . . . . . Ans. $\frac{2}{7}$.
5. $\frac{1369}{10000}$. . . . . Ans. $\frac{37}{100}$.
6. $\frac{18225}{1000000}$. . . . Ans. $\frac{135}{1000}$.

ART. **192.**—A number whose square root can be exactly ascertained, is termed a *perfect square.* Thus, 4, 9, 16, &c., are perfect squares. Comparatively, these numbers are few.

A number whose square root can not be exactly ascertained, is termed an *imperfect square.* Thus, 2, 3, 5, 6, &c., are imperfect squares.

Since the difference of two consecutive square numbers, a^2 and a^2+2a+1, is $2a+1$; therefore, there are always $2a$ imperfect squares between them. Thus, between the square of 4(16), and the square of 5(25), there are $8(2a=2\times4)$ imperfect squares.

A root which can not be exactly expressed, is called a *surd,* or *irrational root.* Thus $\sqrt{2}$ is an irrational root; it is $1.414+$.

The sign $+$, is sometimes placed after an approximate root, to denote that it is less, and the sign $-$, that it is greater than the true root.

It might be supposed, that when the square root of a whole number can not be expressed by a whole number, that it might be found exactly equal to some fraction. We will, therefore, show, that *the square root of an imperfect square, can not be a fraction.*

Let c be an imperfect square, such as 2, and *if possible,* let its square root be equal to a fraction $\frac{a}{b}$, which is supposed to be in its lowest terms.

Then $\sqrt{c}=\frac{a}{b}$; and $c=\frac{a^2}{b^2}$, by squaring both sides.

Now, by supposition, a and b have no common factor, therefore, their squares, a^2 and b^2, can have no common factor, since to square a number, we merely repeat its factors. Consequently, $\frac{a^2}{b^2}$ must be in its lowest terms, and can not be equal to a whole number. Therefore, the equation $c=\frac{a^2}{b^2}$, is not true; and hence, the supposition is false upon which it is founded; that is, that $\sqrt{c}=\frac{a}{b}$; therefore, *the square root of an imperfect square can not be a fraction.*

APPROXIMATE SQUARE ROOTS.

Art. **193.**—To illustrate the method of finding the approximate square root of an imperfect square, let it be required to find the square root of 2 to within $\frac{1}{3}$.

Reducing 2 to a fraction whose denominator is 9 (the square of 3, the denominator of the fraction $\frac{1}{3}$), we have $2=\frac{18}{9}$.

Now, the square root of 18 is greater than 4, and less than 5; therefore, the square root of $\frac{18}{9}$ is greater than $\frac{4}{3}$, and less than $\frac{5}{3}$; therefore, $\frac{4}{3}$ is the square root of 2 to within less than $\frac{1}{3}$.

Hence the

RULE,

FOR EXTRACTING THE SQUARE ROOT OF A WHOLE NUMBER TO WITHIN A GIVEN FRACTION.

Multiply the given number by the square of the denominator of the fraction which determines the degree of approximation; extract the square root of this product to the nearest unit, and divide the result by the denominator of the fraction.

EXAMPLES.

1. Find the square root of 5 to within $\frac{1}{5}$. . . . . . Ans. $2\frac{1}{5}$.
2. Find the square root of 7 to within $\frac{1}{13}$. . . . . Ans. $2\frac{8}{13}$.
3. Find the square root of 15 to within $\frac{1}{23}$. . . . Ans. $3\frac{20}{23}$.
4. Find the square root of 27 to within $\frac{1}{30}$. . . . . Ans. $5\frac{1}{6}$.
5. Find the square root of 14 to within $\frac{1}{10}$. . . . . Ans. 3.7.
6. Find the square root of 15 to within $\frac{1}{100}$. . . . Ans. 3.87.

Since the square of 10 is 100, the square of 100, 10000, and so on, the number of ciphers in the square of the denominator of a decimal fraction is equal to twice the number in the denominator itself. *Therefore, when the fraction which determines the degree of approximation is a decimal, it is merely necessary to add two ciphers for each decimal place required; and, after extracting the root, to point off from the right, one place of decimals for each two ciphers added.*

7. Find the square root of 2 to six places of decimals.
Ans. 1.414213.
8. Find the square root of 5 to five places of decimals.
Ans. 2.23606.

Review.—191. How is the square root of a fraction found, when both terms are perfect squares? 192. When is a number a perfect square? Give examples. When is a number an imperfect square? How can you determine the number of imperfect squares between any two consecutive perfect squares? What is a root called, which can not be exactly expressed? Prove that the square root of an imperfect square can not be a fraction. 193. How do you find the approximate square root of an imperfect square to within any given fraction? What is the rule, when the fraction which determines the degree of approximation, is a decimal?

9. Find the square root of 10. . . . . . . Ans. 3.162277+
10. Find the square root of 101. . . . . . Ans. 10.049875+
11. Find the square root of 60. . . . . . Ans. 7.74596+

ART. **194.**—To find the approximate square root of a fraction.

1. Let it be required to find the square root of $\frac{3}{7}$ to within $\frac{1}{7}$.

$$\tfrac{3}{7}=\tfrac{3}{7}\times\tfrac{7}{7}=\tfrac{21}{49}.$$

Now, since the square root of 21 is greater than 4, and less than 5, therefore, the square root of $\frac{21}{49}$ is greater than $\frac{4}{7}$, and less than $\frac{5}{7}$; hence $\frac{4}{7}$ is the square root of $\frac{3}{7}$ to within less than $\frac{1}{7}$.

Hence, *if we multiply the numerator of a fraction by its denominator, then extract the square root of the product to the nearest unit, and divide the result by the denominator, the quotient will be the square root of the fraction to within one of its equal parts.*

2. Find the square root of $\frac{4}{11}$ to within $\frac{1}{11}$. . . . . Ans. $\frac{6}{11}$.
3. Find the square root of $\frac{7}{15}$ to within $\frac{1}{15}$. . . . . Ans. $\frac{2}{3}$.
4. Find the square root of $\frac{10}{13}$ to within $\frac{1}{13}$. . . . . Ans. $\frac{11}{13}$.

Since any decimal may be written in the form of a fraction having a denominator a perfect square, by adding ciphers to both terms (thus, $.4=\frac{40}{100}=\frac{4000}{10000}$, &c.), therefore, the square root may be found, as in the method of approximating to the square root of a whole number, by *annexing ciphers to the given decimal, until the number of decimal places shall be equal to double the number required in the root. Then, after extracting the root, pointing off from the right, the required number of decimal places.*

Find the square root

5. Of .6 to six places of decimals. . . . . . Ans. .774596.
6. Of .29 to six places of decimals. . . . . . Ans. .538516.

The square root of a whole number and a decimal, may be found in the same manner. Thus, the square root of 2.5 is the same as the square root of $\frac{250}{100}$, which, carried out to 6 places of decimals, is 1.581138+.

7. Find the square root of 10.76 to six places of decimals.
Ans. 3.280243.
8. Find the square root of 1.1025. . . . . Ans. 1.05.

When the denominator of a fraction is a perfect square, its square root may be found by extracting the square root of the numerator to as many places of decimals as are required, and dividing the result by the square root of the denominator. Or, by reducing the fraction to a decimal, and then extracting its square

REVIEW.—194. How do you find the approximate square root of a fraction to within one of the equal parts of the denominator? How do you extract the square root of a decimal? How do you extract the square root of a fraction, when both terms are not perfect squares?

root. When the denominator of the fraction is not a perfect square, the latter method should be used.

9. Find the square root of $\frac{3}{4}$ to five places of decimals.

$$\sqrt{3}=1.73205+,\ \sqrt{4}=2,\ \sqrt{\tfrac{3}{4}}=\tfrac{1.73205+}{2}=.86602+.$$

Or, $\frac{3}{4}=.75$, and $\sqrt{.75}=.86602+$.

10. Find the square root of $3\frac{2}{9}$. . . . . . Ans. 1.795054+.
11. Find the square root of $\frac{7}{16}$. . . . . . Ans. .661437+.
12. Find the square root of $3\frac{1}{4}$. . . . . . Ans. 1.802775+.
13. Find the square root of $5\frac{8}{9}$. . . . . . Ans. 2.426703+.
14. Find the square root of $\frac{1}{7}$. . . . . Ans. .377964+.
15. Find the square root of $\frac{7}{8}$. . . . . Ans. .935414+.
16. Find the square root of $2\frac{1}{3}$. . . . . . Ans. 1.527525+.

EXTRACTION OF THE SQUARE ROOT OF MONOMIALS.

Art. **195.**—From the principles in Art. 179, it is evident, that in order to square a monomial, we must square its coëfficient, and multiply the exponent of each letter by 2. Thus,

$$(3ab^2)^2=3ab^2\times3ab^2=9a^2b^4.$$

Therefore, $\sqrt{9a^2b^4}=3ab^2$. Hence, the

RULE,

FOR EXTRACTING THE SQUARE ROOT OF A MONOMIAL.

Extract the square root of the coëfficient, and divide the exponent of each letter by 2.

Since $+a\times+a=+a^2$, and $-a\times-a=+a^2$,

Therefore $\sqrt{a^2}=+a$, or $-a$.

Hence, the square root of any positive quantity is either *plus*, or *minus*. This is generally expressed, by writing the double sign before the square root. Thus, $\sqrt{4a^2}=\pm2a$, which is read, *plus or minus* $2a$.

If a monomial is *negative*, the extraction of the square root is impossible, since the square of any quantity, either positive or negative, is necessarily positive. Thus, $\sqrt{-9}$, $\sqrt{-4a^2}$, $\sqrt{-b}$, are algebraic symbols, which indicate impossible operations. Such expressions are termed *imaginary quantities.* They occur, in attempting to find the value of the unknown quantity in an equation of the second degree, where some absurdity or impossibility exists in the equation, or in the problem from which it was derived. See Art. 218.

Review.—195. How do we find the square of a monomial? How, then, do we find the square root of a monomial? What is the sign of the square root of any positive quantity? Why is the extraction of the square root of a negative monomial impossible? Give examples of algebraic symbols that indicate impossible operations. What are they termed? Under what circumstances do they occur?

Find the square root of each of the following monomials.

1. $4a^2x^2$. . . . Ans. $\pm 2ax$.
2. $9x^2y^4$. . . Ans. $\pm 3xy^2$.
3. $25a^2b^2c^4$. . Ans. $\pm 5abc^2$.
4. $36a^4b^6x^2$. . Ans. $\pm 6a^2b^3x$.
5. $16m^2n^4y^6$. . Ans. $\pm 4mn^2y^3$.
6. $49a^2b^4c^8$. . Ans. $\pm 7ab^2c^4$.
7. $625x^2z^4$. . . Ans. $\pm 25xz^2$.
8. $1156a^2x^4z^6$. Ans. $\pm 34ax^2z^3$.

Since $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$, therefore, $\sqrt{\frac{a^2}{b^2}} = \pm\frac{a}{b}$; that is, *to find the square root of a monomial fraction, extract the square root of both terms.*

9. Find the square root of $\frac{4a^2}{9b^2}$. . . . . Ans. $\pm\frac{2a}{3b}$.

10. Find the square root of $\frac{16x^2y^4}{25a^2z^2}$. . . . . . . Ans. $\pm\frac{4xy^2}{5az}$.

EXTRACTION OF THE SQUARE ROOT OF POLYNOMIALS.

ART. **196.**—In order to deduce a rule, for extracting the square root of a polynomial, let us first examine the relation that exists between the several terms of any quantity and its square.

$(a+b)^2 = a^2+2ab+b^2 = a^2+(2a+b)b$.

$(a+b+c)^2 = a^2+2ab+b^2+2ac+2bc+c^2 = a^2+(2a+b)b+(2a+2b+c)c$.

$(a+b+c+d)^2 = a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2 = a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d$.

Or, by calling the successive terms of a polynomial r, r', r'', r''', we shall have $(r+r'+r''+r''')^2 = r^2+(2r+r')r'+(2r+2r'+r'')r''+(2r+2r'+2r''+r''')r'''$.

In this formula, r, r', r'', r''', may represent any algebraic quantities whatever, either whole or fractional, positive or negative.

Hence, we see, that the square of any polynomial is formed according to the following *law:*

The square of any polynomial is equal to the square of the first term—plus twice the first term, plus the second, multiplied by the second—plus twice the first and second terms, plus the third, multiplied by the third—plus twice the first, second, and third terms, plus the fourth, multiplied by the fourth, and so on. Hence, by reversing the operation, we have the

RULE,

FOR EXTRACTING THE SQUARE ROOT OF A POLYNOMIAL.

1st. *Arrange the polynomial with reference to a certain letter; then find the first term of the root, by extracting the square root of the first term of the polynomial; place the result on the right, and subtract its square from the given quantity.*

2d. *Divide the first term of the remainder, by double the part of the root already found, and annex the result both to the root and the divisor. Multiply the divisor thus increased, by the second term of the root, and subtract the product from the remainder.*

3d. *Double the terms of the root already found, for a partial divisor, and divide the first term of the remainder, by the first term of the divisor, and annex the result both to the root and the partial divisor. Multiply the divisor thus increased, by the third term of the root, and subtract the product from the last remainder. Then proceed in a similar manner, to find the other terms.*

REMARK.—In the course of the operations on any example, when we find a remainder, of which the first term is not exactly divisible by double the first term of the root, we may conclude that the polynomial is not a perfect square.

EXAMPLES.

1. Find the square root of $r^2+2rr'+r'^2+2rr''+2r'r''+r''^2$

$$
\begin{array}{l|l}
 & r^2+2rr'+r'^2+2rr''+2r'r''+r''^2 \;\big|\; \underline{r+r'+r''}, \text{ root.} \\
 & \underline{r^2} \\
2r+r' & 2rr'+r'^2 \\
 & \underline{2rr'+r'^2} \\
2r+2r'+r'' & 2rr''+2r'r''+r''^2 \\
 & \underline{2rr''+2r'r''+r''^2}
\end{array}
$$

The square root of the first term is r, which we write as the first term of the root. We next subtract the square of r from the given polynomial, and dividing the first term of the remainder $2rr'$, by $2r$, the double of the first term of the root, the quotient is r, the second term of the root. We next place r' in the root, and also in the divisor, and multiply the divisor thus increased, by r', and subtract the product from the first remainder. We then double the terms $r+r'$, of the root already found, for a partial divisor, and find that the quotient of $2rr''$, the first term of the remainder, divided by $2r$, the first term of the divisor, is r'', the third term of the root. Completing the divisor, multiplying by r'', and subtracting, we find there is nothing left.

NOTE.—The first remainder consists of all the terms after r^2, and the second, of all after r'^2. It is useless to bring down more terms than have corresponding terms in the quantity to be subtracted.

REVIEW.—196. What is the square of $a+b$? Of $a+b+c$? Of $a+b+c+d$? Of $r+r'+r''+r'''$? What may r, r', &c., represent? According to what law is the square of any polynomial formed? By reversing this law, what rule do we obtain, for extracting the square root of a polynomial? When may we conclude that a polynomial is not a perfect square?

2. Find the square root of the polynomial $25x^2y^2-24xy^3-12x^3y+4x^4+16y^4$.

Arranging the polynomial with reference to x, we have

$$\begin{array}{r|l}
 & 4x^4-12x^3y+25x^2y^2-24xy^3+16y^4 \quad \underline{|2x^2-3xy+4y^2}, \text{ root.} \\
 & \underline{4x^4} \\
4x^2-3xy & -12x^3y+25x^2y^2 \\
 & \underline{-12x^3y+\ 9x^2y^2} \\
4x^2-6xy+4y^2 & 16x^2y^2-24xy^3+16y^4 \\
 & \underline{16x^2y^2-24xy^3+16y^4}
\end{array}$$

It is easily seen, that the operation is analogous to that of extracting the square root of whole numbers.

Find the square root of the following polynomials.

3. x^2+4x+4. . . . . Ans. $x+2$.
4. $4x^2-12x+9$. . . . . Ans. $2x-3$.
5. $x^2y^2-8xy+16$. . . . . Ans. $xy-4$.
6. $4a^2x^2+25y^2z^2-20axyz$. . . . . Ans. $2ax-5yz$.
7. $x^4+4x^3+6x^2+4x+1$. . . . . Ans. x^2+2x+1.
8. $4x^4-4x^3+13x^2-6x+9$. . . . . Ans. $2x^2-x+3$.
9. $9y^4-12y^3+34y^2-20y+25$. . . . . . Ans. $3y^2-2y+5$.
10. $a^4x^4+6a^2b^2x^2-4a^3bx^3-4ab^3x+b^4$. . . Ans. $a^2x^2-2abx+b^2$.
11. $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$.
    Ans. $1-2x+3x^2-4x^3$.
12. $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$.
    Ans. $a^3-3a^2x+3ax^2-x^3$.
13. $x^2+ax+\frac{1}{4}a^2$. . . . . Ans. $x+\frac{1}{2}a$.
14. $x^2-2x+1+2xy-2y+y^2$. . . . . Ans. $x+y-1$.
15. $x(x+1)(x+2)(x+3)+1$. . . . . Ans. x^2+3x+1.

ART. **197.**—The following remarks will be found useful.

1st. *No binomial can be a perfect square;* for, the square of a monomial is a monomial, and the square of a binomial is a trinomial. Thus, a^2+b^2 is not a perfect square; but if we add to it $2ab$, it becomes the square of $a+b$; and subtracting from it $2ab$, it becomes the square of $a-b$.

2d. In order that a trinomial may be a perfect square, the two extreme terms must be perfect squares, and the middle term the double product of the square roots of the extreme terms. Hence, to obtain the square root of a trinomial when it is a perfect square, *extract the square roots of the two extreme terms, and unite them by the sign plus or minus, according as the second term is plus or minus.*

REVIEW.—197. Why can no binomial be a perfect square? Give an example. What is necessary, in order that a trinomial may be a perfect square? When a trinomial is a perfect square, how may its square root be found? Give an example.

Thus, $4a^2-12ac+9c^2$ is a perfect square, since $\sqrt{4a^2}=2a$, $\sqrt{9c^2}=3c$, and $+2a\times-3c\times2=-12ac$. But $9x^2+12xy+16y^2$, is not a perfect square; since $\sqrt{9x^2}=3x$, $\sqrt{16y^2}=4y$, and $3x\times4y\times2=24xy$, which is not equal to the middle term $12xy$.

RADICALS OF THE SECOND DEGREE.

ART. **198.**—FROM the rule Art. 195, it is evident, that *when a monomial is a perfect square, its numeral coëfficient is a perfect square, and the exponent of each letter is exactly divisible by* 2. Thus, $4a^2$ is a perfect square, while $5a^3$ is not a perfect square, because the coëfficient, 5, is not a perfect square, and the exponent, 3, is not exactly divisible by 2.

When the exact division of the exponent can not be performed, it may be indicated, by writing the divisor under it, in the form of a fraction. Thus, $\sqrt{a^3}$ may be written $a^{\frac{3}{2}}$.

Since a is the same as a^1 the square root of a may be expressed thus, $a^{\frac{1}{2}}$. For this reason, the fractional exponent, $\frac{1}{2}$, is used to indicate the extraction of the square root. Thus, $\sqrt{a^2+2ax+x^2}$ and $(a^2+2ax+x^2)^{\frac{1}{2}}$, also $\sqrt{4}$ and $4^{\frac{1}{2}}$, indicate the same operation; the radical sign, $\sqrt{\ }$, and the fractional exponent, $\frac{1}{2}$, being regarded as equivalent.

Quantities of which the square root can not be exactly ascertained, are termed *radicals of the second degree.* They are also called, *irrational quantities,* or *surds.* Such are the quantities $\sqrt{a}$, $\sqrt{2}$, $a\sqrt{b}$, and $5\sqrt{3}$. Or, as they may be otherwise written, $a^{\frac{1}{2}}$, $2^{\frac{1}{2}}$, $ab^{\frac{1}{2}}$, and $5(3)^{\frac{1}{2}}$. The quantity which stands before the radical sign, is called the *coëfficient* of the radical. Thus, in the expressions $a\sqrt{b}$, and $3\sqrt{5}$, the quantities a and 3 are called coëfficients.

Two radicals are said to be *similar,* when the quantities under the radical sign are the same in both. Thus, $3\sqrt{2}$ and $7\sqrt{2}$ are similar radicals; so, also, are $b\sqrt{a}$ and $2c\sqrt{a}$.

Two radicals that are not similar, may frequently become so, by simplification. This gives rise to

REVIEW.—198. When is a monomial a perfect square? Give an example. How may the square root of a quantity be expressed, without the radical sign? What are radicals of the second degree? What are radicals otherwise called? What is the coëfficient of a radical? When are two radicals similar?

REDUCTION OF RADICALS OF THE SECOND DEGREE.

ART. **199.**—Reduction of radicals of the second degree, consists in changing the form of the quantities without altering their value. It is founded on the following principle.

The square root of the product of two or more factors, is equal to the product of the square roots of those factors.

That is, $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$; which is thus proved:

$$(\sqrt{ab})^2=ab$$

and $(\sqrt{a}\times\sqrt{b})^2=\sqrt{a}\times\sqrt{b}\times\sqrt{a}\times\sqrt{b}=\sqrt{a}\times\sqrt{a}\times\sqrt{b}\times\sqrt{b}=ab.$

Hence, $\sqrt{ab}$ and $\sqrt{a}\times\sqrt{b}$ are equal to each other, since the square of each is equal to ab.

From this principle, we have $\sqrt{36}=\sqrt{4\times9}=2\times3$, $\sqrt{144}=\sqrt{9\times16}=3\times4$.

Any radical of the second degree, can be reduced to a simpler form when it can be separated into factors, one of which is a perfect square.

Thus, $\sqrt{12}=\sqrt{4\times3}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$

$\sqrt{a^3b}=\sqrt{a^2\times ab}=\sqrt{a^2}\times\sqrt{ab}=a\sqrt{ab}$

$\sqrt{27a^3c^4}=\sqrt{9a^2c^4\times3a}=\sqrt{9a^2c^4}\times\sqrt{3a}=3ac^2\sqrt{3a}.$

From the preceding illustrations, we derive the

RULE,

FOR REDUCING A RADICAL OF THE SECOND DEGREE TO ITS SIMPLEST FORM.

1st. *Separate the quantity to be reduced, into two parts, one of which shall contain all the factors that are perfect squares, and the other the remaining factors.*

2d. *Extract the square root of the part that is a perfect square, and prefix it as a coëfficient, to the other part placed under the radical sign.*

To determine if any quantity contains a numeral factor that is a perfect square, ascertain if it is exactly divisible by either of the perfect squares, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, &c. If not thus divisible, it contains no factor that is a perfect square, and the numerical factor can not be reduced.

Reduce each of the following radicals to its simplest form.

1.	$\sqrt{8a^2}$.	Ans. $2a\sqrt{2}$.	9.	$\sqrt{32a^6b^2c^4}$.	Ans. $4a^3bc^2\sqrt{2}$.
2.	$\sqrt{12a^3}$.	Ans. $2a\sqrt{3a}$.	10.	$\sqrt{40a^2b^3c^5}$.	A. $2abc^2\sqrt{10bc}$.
3.	$\sqrt{16a^3b}$.	Ans. $4a\sqrt{ab}$.	11.	$\sqrt{44a^5b^3c}$.	A. $2a^2b\sqrt{11abc}$.
4.	$\sqrt{18a^4b^3c^3}$.	A. $3a^2bc\sqrt{2bc}$.	12.	$\sqrt{45a^4b^6c^4}$.	Ans. $3a^2b^3c^2\sqrt{5}$.
5.	$\sqrt{20a^3b^3c^3}$.	A. $2abc\sqrt{5abc}$.	13.	$\sqrt{48a^8b^6c^4}$.	A. $4a^4b^3c^2\sqrt{3}$.
6.	$3\sqrt{24a^4c^2}$.	Ans. $6a^2c\sqrt{6}$.	14.	$\sqrt{75a^3b^3c^3}$.	A. $5abc\sqrt{3abc}$.
7	$4\sqrt{27a^3c^3}$.	A. $12ac\sqrt{3ac}$.	15.	$\sqrt{128a^6b^4c^2}$.	A. $8a^3b^2c\sqrt{2}$.
8.	$7\sqrt{28a^5c^2}$.	A. $14a^2c\sqrt{7a}$.	16.	$\sqrt{243a^3b^2c}$.	A. $9ab\sqrt{3ac}$.

In a similar manner, polynomials may sometimes be simplified. Thus, $\sqrt{(2a^3-4a^2b+2ab^2)}=\sqrt{2a(a^2-2ab+b^2)}=(a-b)\sqrt{2a}$.

A fractional radical of the second degree may be reduced to its simplest form, by the same rule, by first multiplying both terms by any quantity that will render the denominator a perfect square; separating the fraction into two factors, one of which is a perfect square, then extracting the square root of the square factor, and placing it before the other factor placed under the radical sign.

17. Reduce $\sqrt{\frac{2}{3}}$ to its simplest form.

$$\sqrt{\tfrac{2}{3}}=\sqrt{\tfrac{2}{3}\times\tfrac{3}{3}}=\sqrt{\tfrac{6}{9}}=\sqrt{\tfrac{1}{9}\times 6}=\sqrt{\tfrac{1}{9}}\times\sqrt{6}=\tfrac{1}{3}\sqrt{6}. \quad \text{Ans.}$$

Reduce the following fractional radicals to their simplest forms.

18. $\sqrt{\frac{3}{5}}$. Ans. $\frac{1}{5}\sqrt{15}$.
19. $\sqrt{\frac{7}{8}}$. Ans. $\frac{1}{4}\sqrt{14}$.
20. $\sqrt{\frac{12}{25}}$. Ans. $\frac{2}{5}\sqrt{3}$.
21. $\sqrt{\frac{11}{18}}$. Ans. $\frac{1}{6}\sqrt{22}$.
22. $9\sqrt{\frac{16}{27}}$. Ans. $4\sqrt{3}$.
23. $5\sqrt{\frac{9}{10}}$. Ans. $\frac{3}{2}\sqrt{10}$.
24. $10\sqrt{\frac{3}{50}}$. Ans. $\sqrt{6}$.
25. $7\sqrt{\frac{3}{28}}$. Ans. $\frac{1}{2}\sqrt{21}$.

Since $a=\sqrt{a^2}$, and $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{4\times3}=\sqrt{12}$, it is obvious, that any quantity may be reduced to the form of a radical of the second degree, by squaring it, and placing it under the radical sign. By the same principle, the coëfficient of a radical may be passed under the radical sign.

26. Reduce 5 to the form of a radical of the second degree.
Ans. $\sqrt{25}$.

27. Reduce $2a$ to the form of a radical of the second degree.
Ans. $\sqrt{4a^2}$.

28. Express the quantity $3\sqrt{5}$, entirely under the radical.
Ans. $\sqrt{45}$.

29. Pass the coëfficient of the quantity $3c\sqrt{2c}$, under the radical.
Ans. $\sqrt{18c^3}$.

30. Pass the coëfficient of the quantity $5\sqrt{3}$, under the radical.
Ans. $\sqrt{75}$

ADDITION OF RADICALS OF THE SECOND DEGREE.

ART. **200.**—1. What is the sum of $3\sqrt{2}$ and $5\sqrt{2}$?

It is evident, that 3 times and 5 times any certain quantity must make 8 times that quantity, therefore

$$3\sqrt{2}+5\sqrt{2}=8\sqrt{2}.$$

In the same manner, $\sqrt{2}+\sqrt{8}=\sqrt{2}+2\sqrt{2}=3\sqrt{2}$.

2. What is the sum of $2\sqrt{3}$ and $5\sqrt{7}$?

Since dissimilar quantities can not be collected into one sum, we can only add these expressions by placing the sign of addition between them; that is, the sum of $2\sqrt{3}$ and $5\sqrt{7}=2\sqrt{3}+5\sqrt{7}$.

Hence, the

RULE,

FOR THE ADDITION OF RADICALS OF THE SECOND DEGREE.

1st. *Reduce the radicals to their simplest form.*

2d. *Then, if the radicals are similar, prefix the sum of their coëfficients to the common radical; but, if they are not similar, connect them by their proper signs.*

Find the sum of the radicals in each of the following examples.

3. $\sqrt{8}$ and $\sqrt{18}$. . . . . Ans. $5\sqrt{2}$.
4. $\sqrt{12}$ and $\sqrt{27}$. . . . . Ans. $5\sqrt{3}$.
5. $\sqrt{20}$ and $\sqrt{80}$. . . . . Ans. $6\sqrt{5}$.
6. $\sqrt{24}$ and $\sqrt{150}$. . . . . Ans. $7\sqrt{6}$.
7. $\sqrt{8}$, $\sqrt{32}$, and $\sqrt{50}$. . . . . Ans. $11\sqrt{2}$.
8. $\sqrt{40}$, $\sqrt{90}$, and $\sqrt{250}$. . . . . Ans. $10\sqrt{10}$.
9. $\sqrt{28a^2b^2}$ and $\sqrt{112a^2b^2}$. . . . . Ans. $6ab\sqrt{7}$.
10. $\sqrt{75a^2c}$, and $\sqrt{147a^2c}$. . . . . Ans. $12a\sqrt{3c}$.
11. $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{3}{25}}$. . . . . Ans. $\frac{8}{15}\sqrt{3}$.
12. $\sqrt{\frac{1}{5}}$ and $\sqrt{\frac{5}{49}}$. . . . . Ans. $\frac{12}{35}\sqrt{5}$.
13. $\sqrt{\frac{1}{2}}$ and $\sqrt{8}$. . . . . Ans. $\frac{5}{2}\sqrt{2}$.
14. $2\sqrt{\frac{3}{4}}$ and $3\sqrt{12}$. . . . . Ans. $7\sqrt{3}$.
15. $\frac{1}{2}\sqrt{\frac{1}{2}}$ and $\frac{3}{4}\sqrt{2}$. . . . . Ans. $\sqrt{2}$.
16. $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$. . . . . Ans. $\frac{31}{10}\sqrt{6}$.
17. $\sqrt{48a^2c^2x}$ and $\sqrt{12b^2x}$. . . . . . . Ans. $(4ac+2b)\sqrt{3x}$.
18. Find the sum of $\sqrt{(2a^3-4a^2c+2ac^2)}$ and $\sqrt{(2a^3+4a^2c+2ac^2)}$. Ans. $2a\sqrt{2a}$.
19. Find the sum of $\sqrt{a+x}+\sqrt{ax^2+x^3}+\sqrt{(a+x)^3}$.
Ans. $(1+a+2x)\sqrt{a+x}$.

SUBTRACTION OF RADICALS OF THE SECOND DEGREE.

ART. **201.**—1. Take $3\sqrt{2}$ from $5\sqrt{2}$.

It is evident that 5 times any quantity minus 3 times the quantity, will be equal to 2 times the quantity, therefore

$$5\sqrt{2}-3\sqrt{2}=2\sqrt{2}.$$

In the same manner, $\sqrt{8}-\sqrt{2}=2\sqrt{2}-\sqrt{2}=\sqrt{2}$.

REVIEW.—199. In what does reduction of radicals of the second degree consist? On what principle is it founded? Prove this principle. What is the rule for the reduction of a radical of the second degree to its simplest form? How do you determine if any quantity contains a numerical factor that is a perfect square? How may a fractional radical of the second degree be reduced to its simplest form? 200. What is the rule for the addition of radicals of the second degree?

If the radicals are dissimilar, it is obvious that their difference can only be indicated. Thus, if it be required to take $3\sqrt{a}$ from $5\sqrt{b}$, the difference would be expressed by $5\sqrt{b}-3\sqrt{a}$.

From these illustrations, we derive the

RULE,

FOR THE SUBTRACTION OF RADICALS OF THE SECOND DEGREE.

1st. *Reduce the radicals to their simplest form; then subtract their coëfficients, and prefix the difference to the common radical.*

2d. *If the radicals are not similar, indicate their difference by the proper sign.*

EXAMPLES.

2. $\sqrt{18}-\sqrt{2}$. . . . . Ans. $2\sqrt{2}$.
3. $\sqrt{45a^2}-\sqrt{5a^2}$. . . . . Ans. $2a\sqrt{5}$.
4. $\sqrt{54b}-\sqrt{6b}$. . . . . Ans. $2\sqrt{6b}$.
5. $\sqrt{112a^2c^2}-\sqrt{28a^2c^2}$. . . . . Ans. $2ac\sqrt{7}$.
6. $\sqrt{27b^3c^3}-\sqrt{12b^3c^3}$. . . . . Ans. $bc\sqrt{3bc}$.
7. $\sqrt{36a^5}-\sqrt{4a^5}$. . . . . Ans. $4a^2\sqrt{a}$.
8. $\sqrt{49ab^3c^2}-\sqrt{25ab^3c^2}$. . . . . Ans. $2bc\sqrt{ab}$.
9. $\sqrt{160a^3b^3c}-\sqrt{10a^3b^3c}$. . . . . Ans. $3ab\sqrt{10abc}$.
10. $5a\sqrt{27}-3a\sqrt{48}$. . . . . Ans. $3a\sqrt{3}$.
11. $2\sqrt{\frac{3}{4}}-3\sqrt{\frac{1}{3}}$. . . . . Ans. 0.
12. $\sqrt{\frac{5}{6}}-\sqrt{\frac{10}{27}}$. . . . . Ans. $\frac{1}{18}\sqrt{30}$.
13. $\sqrt{12}-\sqrt{\frac{3}{4}}$. . . . . Ans. $\frac{3}{2}\sqrt{3}$.
14. $3\sqrt{\frac{1}{2}}-\sqrt{2}$. . . . . Ans. $\frac{1}{2}\sqrt{2}$.
15. $\sqrt{\frac{2}{3}}-\sqrt{\frac{2}{27}}$. . . . . Ans. $\frac{2}{9}\sqrt{6}$.
16. From $\sqrt{4a^2x}$ take $a\sqrt{x^3}$ . . . . Ans. $(2a-ax)\sqrt{x}$.
17. From $\sqrt{3m^2x+6mnx+3n^2x}$ take $\sqrt{3m^2x-6mnx+3n^2x}$.
Ans. $2n\sqrt{3x}$.

MULTIPLICATION OF RADICALS OF THE SECOND DEGREE.

ART. **202.**—Since $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$, therefore $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$. See Art. 199.

Also, $a\sqrt{b}\times c\sqrt{d}=a\times c\times\sqrt{b}\times\sqrt{d}=ac\sqrt{bd}$.

From which we have the

RULE,

FOR THE MULTIPLICATION OF RADICALS OF THE SECOND DEGREE.

1st. *Multiply the quantities under the radical sign together, and place the result under the radical.*

2d. *If the radicals have coëfficients, place their product as a coëfficient before the radical sign.*

EXAMPLES.

1. Find the product of $\sqrt{6}$ and $\sqrt{8}$.

$\sqrt{6}\times\sqrt{8}=\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}$. Ans.

2. Find the product of $2\sqrt{14}$ and $3\sqrt{2}$.

$2\sqrt{14}\times3\sqrt{2}=6\sqrt{28}=6\sqrt{4\times7}=6\times2\sqrt{7}=12\sqrt{7}$. Ans.

3. Find the product of $\sqrt{8}$ and $\sqrt{2}$. . . . . Ans. 4.
4. Find the product of $2\sqrt{a}$ and $3\sqrt{a}$. . . . . Ans. $6a$.
5. Find the product of $\sqrt{27}$ and $\sqrt{3}$. . . . . Ans. 9.
6. Find the product of $3\sqrt{2}$ and $2\sqrt{3}$. . . . . . . Ans. $6\sqrt{6}$.
7. Find the product of $3\sqrt{3}$ and $2\sqrt{3}$. . . . . . . Ans. 18.
8. Find the product of $\sqrt{6}$ and $\sqrt{15}$. . . . . Ans. $3\sqrt{10}$.
9. Find the product of $2\sqrt{15}$ and $3\sqrt{35}$. . . . Ans. $30\sqrt{21}$.
10. Find the product of $\sqrt{a^3b^5c}$ and $\sqrt{abc}$. . . . . Ans. a^2b^3c.
11. Find the product of $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{3}{4}}$. . . . . Ans. $\frac{1}{2}$.
12. Find the product of $\sqrt{\frac{2}{5}}$ and $\sqrt{\frac{8}{9}}$. . . . . . Ans. $\frac{4}{15}\sqrt{5}$.
13. Find the product of $2\sqrt{\frac{a}{5}}$ and $3\sqrt{\frac{a}{10}}$. . . . Ans. $\frac{3a}{5}\sqrt{2}$.

When two polynomials contain radicals of the second degree, they may be multiplied together, in the same manner as in multiplication of polynomials, Art. 72, attending, at the same time, to the directions contained in the preceding rule.

14. Find the product of $2+\sqrt{2}$ and $2-\sqrt{2}$. . . . . Ans. 2.
15. Find the product of $1+\sqrt{2}$ and $1-\sqrt{2}$. . . . Ans. -1.
16. Find the product of $\sqrt{x+2}$ by $\sqrt{x-2}$. . . Ans. $\sqrt{x^2-4}$.
17. Find the product of $\sqrt{a+x}$ by $\sqrt{a+x}$. . . . . Ans. $a+x$.
18. Find the product of $\sqrt{ab+bx}$ by $\sqrt{ab-bx}$. A. $\sqrt{a^2b^2-b^2x^2}$.
19. Find the product of $\sqrt{x+2}$ by $\sqrt{x+3}$. Ans. $\sqrt{x^2+5x+6}$.

Perform the operations indicated in the following examples.

20. $(c\sqrt{a}+d\sqrt{b})\times(c\sqrt{a}-d\sqrt{b})$. . . . . Ans. c^2a-d^2b.
21. $(7+2\sqrt{6})\times(9-5\sqrt{6})$. . . . . Ans. $3-17\sqrt{6}$.
22. $(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}-\sqrt{a-x})$. . . . . . . Ans. $2x$.
23. $(x+2\sqrt{ax}+a)(x-2\sqrt{ax}+a)$. . . . . . Ans. $x^2-2ax+a^2$.
24. $(x^2-x\sqrt{2}+1)(x^2+x\sqrt{2}+1)$. . . . . Ans. x^4+1.

DIVISION OF RADICALS OF THE SECOND DEGREE.

Art. 203.—Since Division is the reverse of Multiplication, and since $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$, therefore $\sqrt{ab}\div\sqrt{a}=\sqrt{\frac{ab}{a}}=\sqrt{b}$.

Review.—201. What is the rule for the subtraction of radicals of the second degree? 202. What is the rule for the multiplication of radicals of the second degree? On what principle does it depend?

Also, since $a\sqrt{b}\times c\sqrt{d}=ac\sqrt{bd}$, therefore $ac\sqrt{bd}\div a\sqrt{b}=\frac{ac\sqrt{bd}}{a\sqrt{b}}$ $=\frac{ac}{a}\sqrt{\frac{bd}{b}}=c\sqrt{d}$. Hence, the

RULE,

FOR THE DIVISION OF RADICALS OF THE SECOND DEGREE.

1st. *Find the quotient of the parts under the radical, and place it under the common radical.*

2d. *If the radicals have coëfficients, divide the coëfficient of the dividend by that of the divisor, and prefix the result to the common radical.*

NOTE.—When a radical quantity has no coëfficient prefixed, its coëfficient is understood to be 1. Thus, $\sqrt{2}$ is the same as $1\sqrt{2}$. See Art. 32.

EXAMPLES.

1. Divide $8\sqrt{72}$ by $2\sqrt{6}$.

$$\frac{8\sqrt{72}}{2\sqrt{6}}=\tfrac{8}{2}\sqrt{\tfrac{72}{6}}=4\sqrt{12}=4\sqrt{4\times3}=8\sqrt{3}.\ \text{Ans.}$$

2. Divide $\sqrt{54}$ by $\sqrt{6}$. . . . . Ans. 3.
3. Divide $6\sqrt{54}$ by $3\sqrt{27}$. . . . . Ans. $2\sqrt{2}$.
4. Divide $6\sqrt{28}$ by $2\sqrt{7}$. . . . . Ans. 6.
5. Divide $\sqrt{160}$ by $\sqrt{8}$. . . . . Ans. $2\sqrt{5}$.
6. Divide $15\sqrt{378}$ by $5\sqrt{6}$. . . . . Ans. $9\sqrt{7}$.
7. Divide $\sqrt{a^3}$ by $\sqrt{a}$. . . . . Ans. a.
8. Divide $ab\sqrt{a^3b^3}$ by $b\sqrt{ab}$. . . . . Ans. a^2b.
9. Divide a by $\sqrt{a}$. . . . . Ans. $\sqrt{a}$.
10. Divide $a\sqrt{b}$ by $c\sqrt{d}$. . . . . Ans. $\frac{a}{c}\sqrt{\frac{b}{d}}$, or $\frac{a}{cd}\sqrt{bd}$.
11. Divide $\sqrt{\frac{a}{b}}$ by $\sqrt{\frac{d}{c}}$. . . . . . . Ans. $\sqrt{\frac{ac}{bd}}$, or $\frac{1}{bd}\sqrt{abcd}$.
12. Divide $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{1}{3}}$. . . . . Ans. $\frac{1}{2}\sqrt{6}$.
13. Divide $\sqrt{\frac{3}{4}}$ by $\sqrt{\frac{1}{3}}$. . . . . Ans. $1\frac{1}{2}$.
14. Divide $\frac{2}{3}\sqrt{18}$ by $\frac{1}{2}\sqrt{2}$. . . . . Ans. 4.
15. Divide $\frac{3}{5}\sqrt{\frac{1}{3}}$ by $\frac{1}{2}\sqrt{\frac{3}{5}}$. . . . . Ans. $\frac{2}{5}\sqrt{5}$.
16. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2}+3\sqrt{\frac{1}{2}}$. . . . . Ans. $\frac{1}{10}$.

ART. **204.**—To reduce a fraction whose denominator is either a monomial or a binomial containing radicals of the second degree, to an equivalent fraction having a rational denominator.

REVIEW.—203. What is the rule for the division of radicals of the second degree? On what principle does it depend?

When the fraction is of the form $\frac{a}{\sqrt{b}}$, if we multiply both terms by $\sqrt{b}$, the denominator will become rational. Thus,

$$\frac{a}{\sqrt{b}}=\frac{a\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}}=\frac{a\sqrt{b}}{b}.$$

Since the sum of two quantities, multiplied by their difference, is equal to the difference of their squares; if the fraction is of the form $\frac{a}{b+\sqrt{c}}$, and we multiply both terms by $b-\sqrt{c}$, the denominator will be made rational, since it will be b^2-c. Thus,

$$\frac{a}{b+\sqrt{c}}\times\frac{b-\sqrt{c}}{b-\sqrt{c}}=\frac{ab-a\sqrt{c}}{b^2-c}.$$

For the same reason, if the denominator is $b-\sqrt{c}$, the multiplier will be $b+\sqrt{c}$. If the denominator is $\sqrt{b}+\sqrt{c}$, the multiplier will be $\sqrt{b}-\sqrt{c}$; and, if the denominator is $\sqrt{b}-\sqrt{c}$, the multiplier will be $\sqrt{b}+\sqrt{c}$.

These different forms may be embraced in the following

GENERAL RULE.

If the denominator is a monomial, multiply both terms by the radical quantity; but, if it is a binomial, multiply both terms by the given binomial with the sign of one of its terms changed, and the denominator will be rational.

Reduce the following fractions to equivalent fractions, having rational denominators.

1. $\frac{1}{\sqrt{2}}$. Ans. $\frac{\sqrt{2}}{2}=\frac{1}{2}\sqrt{2}$.
2. $\frac{\sqrt{2}}{\sqrt{3}}$. Ans. $\frac{\sqrt{6}}{3}=\frac{1}{3}\sqrt{6}$.
3. $\frac{1}{2+\sqrt{3}}$. Ans. $2-\sqrt{3}$.
4. $\frac{3}{6-\sqrt{3}}$. Ans. $\frac{1}{11}(6+\sqrt{3})$.
5. $\frac{5}{\sqrt{7}+\sqrt{6}}$. A. $5(\sqrt{7}-\sqrt{6})$.
6. $\frac{8}{\sqrt{5}-\sqrt{3}}$. A. $4(\sqrt{5}+\sqrt{3})$.

REMARK.—The utility of these transformations, consists in diminishing the amount of calculation, necessary to obtain the numerical value of a fractional radical to any required degree of accuracy.

Thus, suppose it is required to obtain the numerical value of the fraction $\frac{1}{\sqrt{2}}$, true to six places of decimals.

If we make the calculation without rendering the denominator rational, it will be found, that we must first extract the square root of 2, to seven

REVIEW.—204. When the denominator of a fraction is either a monomial or a binomial, containing radicals of the second degree, how may it be reduced to a fraction having a rational denominator?

places of decimals, and then divide 1 by this result. But if we render the donominator rational, the calculation merely consists in finding the square root of 2, and then dividing by 2. The work by the latter method, requires only about half the labor of that by the former. Besides, the operator feels certain, if he has made no mistake, that the last figure of his result is correct. Whereas, by the other mode, as the divisor is too small, the quotient figures soon become too large. Thus in this example, if we use seven decimals for a divisor, the seventh figure of the quotient is too large; if we only use six places of decimals, the sixth figure will be erroneous.

7. Find the numerical value of the fraction $\frac{3}{\sqrt{5}}$.

Ans. 1.3416407+.

8. Find the numerical value of the fraction $\frac{3}{\sqrt{5}-\sqrt{2}}$.

Ans. 3.650281+.

9. Find the numerical value of the fraction $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}$.

Ans. 2.805883+.

REMARK.—It is proper to notice, that the signs $\sqrt{\ }$ and $\sqrt{\quad\quad}$, when applied to a *monomial*, both have the same meaning. There is a want of uniformity among the best writers, in the manner of making the radical sign before a monomial.

SIMPLE EQUATIONS CONTAINING RADICALS OF THE SECOND DEGREE.

NOTE TO TEACHERS.—This part of the subject of Equations of the First Degree, could not be treated till after Radicals. It may be omitted entirely by the younger class of pupils.

ART. **205.**—In the solution of questions involving radicals, much will depend on the judgment of the pupil; but the easiest processes can only be learned from *practice*, as almost every question can be solved in several ways.

The following directions will be frequently found useful.

1st. When the equation contains one radical expression, transpose it to one side of the equation, and the rational terms to the other side; then involve both sides to a power corresponding to the radical sign.

Thus, if we have the equation $\sqrt{(x-1)}-1=2$, to find x.
Transposing, $\sqrt{(x-1)}=3$
Squaring, $x-1=9$, from which $x=10$.

2d. When more than one expression is under the radical sign, the operation must be repeated.

Thus, $a+x=\sqrt{(a^2+x\sqrt{c^2+x^2})}$, to find x.
Squaring, $a^2+2ax+x^2=a^2+x\sqrt{c^2+x^2}$.
Reducing and dividing by x, $2a+x=\sqrt{c^2+x^2}$.
Squaring, $4a^2+4ax+x^2=c^2+x^2$.
whence $x=\dfrac{c^2-4a^2}{4a}$.

3d. When there are two radical expressions, it is generally better to make one of them stand alone on one side, before squaring.

Thus, $\sqrt{(x-5)}-3=4-\sqrt{(x-12)}$, to find x.
Transposing, $\sqrt{(x-5)}=7-\sqrt{(x-12)}$.
Squaring, $x-5=49-14\sqrt{(x-12)}+x-12$.
Reducing and transposing, $14\sqrt{(x-12)}=42$.
Dividing, $\sqrt{(x-12)}=3$.
Squaring, $x-12=9$, from which $x=21$.

EXAMPLES FOR PRACTICE.

1. $\sqrt{(x+3)}+3=7$. . . . . Ans. $x=13$.
2. $x+\sqrt{(x^2+11)}=11$. . . . . Ans. $x=5$.
3. $\sqrt{(6+\sqrt{x-1})}=3$. . . . . Ans. $x=10$.
4. $\sqrt{x(a+x)}=a-x$. . . . . Ans. $x=\dfrac{a}{3}$.
5. $\sqrt{x}-2=\sqrt{(x-8)}$. . . . . Ans. $x=9$.
6. $x+\sqrt{x^2-7}=7$. . . . . Ans. $x=4$.
7. $2+\sqrt{3x}=\sqrt{5x+4}$. . . . . Ans. $x=12$.
8. $\sqrt{x+7}=6-\sqrt{x-5}$. . . . . Ans. $x=9$.
9. $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$. . . . . Ans. $x=\dfrac{25a}{16}$.
10. $\sqrt{x+225}-\sqrt{x-424}-11=0$. . . . . . Ans. $x=1000$.
11. $x+\sqrt{2ax+x^2}=a$. . . . . Ans. $x=\frac{1}{4}a$.
12. $\sqrt{x+a}-\sqrt{x-a}=\sqrt{a}$. . . . . Ans. $x=\dfrac{5a}{4}$.
13. $\sqrt{x+12}=2+\sqrt{x}$. . . . . Ans. $x=4$.
14. $\sqrt{8+x}=2\sqrt{1+x}-\sqrt{x}$. . . . . Ans. $x=\frac{1}{3}$.
15. $\sqrt{5x}+\dfrac{12}{\sqrt{5x+6}}=\sqrt{5x+6}$. . . . . Ans. $x=\frac{2}{5}$.
16. $\sqrt{x}-4=\dfrac{237-10x}{4+\sqrt{x}}$. . . . . Ans. $x=23$
17. $\sqrt{x^2+\sqrt{4x^2+x+\sqrt{9x^2+12x}}}=1+x$. . . . . . Ans. $x=\frac{1}{6}$.
18. $\sqrt{a+\sqrt{ax}}=\sqrt{a}-\sqrt{a-\sqrt{ax}}$. . . . . . . Ans. $x=\frac{3}{4}a$.

19 $b(\sqrt{x}+\sqrt{b})=a(\sqrt{x}-\sqrt{b})$. . . . . Ans. $x=\frac{b(a+b)^2}{(a-b)^2}$.

20 $\sqrt{x}+\sqrt{ax}=a-1$. . . . . Ans. $x=(\sqrt{a}-1)^2$.

CHAPTER VII.

EQUATIONS OF THE SECOND DEGREE.

ART. **206.**—An Equation of the Second Degree (See Art. 148), is one in which the greatest exponent of the unknown quantity is 2. Thus, $x^2=9$, and $5x^2+3x=26$, are equations of the second degree.

An equation containing two or more unknown quantities, in which the greatest exponent, or the greatest sum of the exponents of the unknown quantities, is 2, is also an equation of the second degree. Thus, $xy=6$, $x^2+xy=8$, $xy+x+y=11$, are equations of the second degree.

Equations of the Second Degree, are frequently denominated *Quadratic Equations.*

ART. **207.**—Equations of the second degree are of two kinds—*incomplete* and *complete.*

An incomplete equation of the second degree, is of the form $ax^2=b$, and contains only the second power of the unknown quantity, and known terms. Thus, $x^2=9$, and $8x^2-5x^2=12$, are incomplete equations of the second degree.

An incomplete equation of the second degree, is frequently denominated a *pure quadratic* equation.

A complete equation of the second degree, is of the form $ax^2+bx=c$, and contains both the first and second powers of the unknown quantity, and known terms. Thus, $3x^2+4x=20$, and $ax^2-bx^2+dx-ex=f-g$, are complete equations of the second degree.

A complete equation of the second degree, is frequently denominated an *affected quadratic* equation.

REVIEW.—206. What is an equation of the second degree? Give examples. If an equation contains two unknown quantities, when is it of the second degree? Give examples. 207. How many kinds of equations of the second degree are there? What are they? What is the form of an incomplete equation of the second degree? What does it contain? Give an example. What is the form of a complete equation of the second degree? What does it contain? Give an example. What is a pure quadratic equation? What is an affected quadratic equation?

ART. **208.**—Every equation of the second degree, may be reduced to one of the forms $ax^2=b$, or $ax^2+bx=c$. For, in an incomplete equation, all the terms containing x^2 may be collected together, and then, if the coëfficient of x^2 contains more than one term, it may be assumed equal to a single quantity, as a, and the sum of the known quantities, to another quantity, b, and then the equation becomes $ax^2=b$, or $ax^2-b=0$.

So a complete equation may be similarly reduced; for all the terms containing x^2, may be reduced to one term, as ax^2; and those containing x, to one, as bx; and the known terms to one, as c; then the equation is $ax^2+bx=c$, or $ax^2+bx-c=0$.

Hence, we infer: *That every equation of the second degree, may be reduced to an incomplete equation involving two terms, or to a complete equation involving three terms.*

Frequent illustrations of these principles will occur hereafter.

INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

ART. **209.**—1. Let it be required to find the value of x in the equation $x^2-16=0$.

Transposing, $x^2=16$

Extracting the square root of both members,

$$x=\pm4, \text{ that is, } x=+4, \text{ or } -4.$$

Verification. $(+4)^2-16=16-16=0$.

or, $(-4)^2-16=16-16=0$.

2. Find the value of x in the equation $5x^2+4=49$.

Transposing, $5x^2=45$

Dividing, $x^2=9$

Extracting the square root of both sides,

$$x=\pm3.$$

3. Find the value of x in the equation $\frac{2x^2}{3}+\frac{3x^2}{4}=5\frac{2}{3}$.

Clearing of fractions, $8x^2+9x^2=68$

Reducing, $17x^2=68$

Dividing, $x^2=4$

Extracting the square root, $x=\pm2$.

4. Given $ax^2+b=cx^2+d$, to find the value of x.

$$ax^2-cx^2=d-b$$

$$\text{or, } (a-c)x^2=d-b$$

$$x^2=\frac{d-b}{a-c}$$

$$x=\pm\sqrt{\frac{d-b}{a-c}}.$$

From the preceding examples, we derive the

RULE,

FOR THE SOLUTION OF AN INCOMPLETE EQUATION OF THE SECOND DEGREE.

Reduce the equation to the form $ax^2=b$. Divide both sides by the coëfficient of x^2, and then extract the square root of both members

ART. **210.**—If we take the equation $ax^2=b$

we have $$x^2=\frac{b}{a}$$

and $$x=\pm\sqrt{\frac{b}{a}};\text{ that is,}$$

$$x=+\sqrt{\frac{b}{a}},\text{ and }x=-\sqrt{\frac{b}{a}}.$$

If we assume $\frac{b}{a}=m^2$, then $x^2=m^2$

By transposing, $x^2-m^2=0$

By separating into factors, $(x+m)(x-m)=0.$

Now, this equation can be satisfied in two ways, and in two *only;* that is, by making either of the factors equal to 0.

By making the second factor equal to 0, we have

$$x-m=0,\text{ or }x=+m.$$

By making the first factor equal to 0, we have

$$x+m=0,\text{ or }x=-m.$$

Since the equation $(x+m)(x-m)=0$, can be satisfied *only* in these two ways, it follows, that the values of x obtained from these conditions, are the only values of the unknown quantity.

Hence we conclude

1st. *That every incomplete equation of the second degree, has two roots, and only two.*

2d. *That these roots are equal, but have contrary signs.*

Find the roots of the equation, or the values of x, in each of the following examples.

1. $x^2-8=28$. . . . . Ans. $x=\pm6$.
2. $3x^2-15=83+x^2$. . . . . Ans. $x=\pm7$.
3. $a^2x^2-b^2=0$. . . . . Ans. $x=\pm\frac{b}{a}$.
4. $7x^2-25=4x^2-13$. . . . . Ans. $x=\pm2$.

REVIEW.—208. To what two forms may every equation of the second degree be reduced? Why? 209. What is the rule for the solution of an incomplete equation of the second degree? 210. Show that every incomplete equation of the second degree, has two roots, and only two; and that those roots are equal, but have contrary signs.

5. $5x^2-2=8-35x^2$. . . . . Ans. $x=\pm\frac{1}{2}$.

6. $\frac{1}{3}x^2-1=\frac{4x^2}{27}+\frac{2}{3}$. . . . . Ans. $x=\pm3$.

7. $\frac{5x^2}{3}+12=\frac{8x^2}{7}+37\frac{2}{3}$. . . . . Ans. $x=\pm7$.

8. $(2x-5)^2=x^2-20x+73$. . . . . Ans. $x=\pm4$.

9. $ax^2-b=(a-b)x^2+c$. . . . . Ans. $x=\pm\sqrt{\frac{b+c}{b}}$.

10. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{10a^2}{x^2-a^2}$. . . . . Ans. $x=\pm2a$.

11. $\frac{x-a}{a}-\frac{a-2x}{x-a}=\frac{x^2+bx}{x^2-a^2}$. . . . . Ans. $x=\pm\sqrt{ab}$.

QUESTIONS PRODUCING INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

ART. **211.**—In the solution of a problem producing an equation containing the second power of the unknown quantity, the equation is found on the same principle, as in questions producing equations of the first degree. See Art. 156.

1. Find a number, whose $\frac{2}{3}$ multiplied by its $\frac{2}{5}$, will be equal to 60.

Let $x=$ the number; then $\frac{2x}{3}\times\frac{2x}{5}=\frac{4x^2}{15}=60$

$$4x^2=900$$
$$x^2=225$$
$$x=15$$

2. What number is that, of which the product of its third and fourth parts is equal to 108? Ans. 36.

3. What number is that, whose square diminished by 16, is equal to half its square increased by 16? Ans. 8.

4. What number is that, whose square diminished by 54, is equal to the square of its half, increased by 54? Ans. 12.

5. What number is that, which being divided by 9, gives the same quotient, as 16 divided by the number? Ans. 12.

6. What two numbers are to each other as 3 to 5, and the difference of whose squares is 64?

Let $3x=$ the less number; then $5x=$ the greater.

And $(5x)^2-(3x)^2=64$

Or $25x^2-9x^2=16x^2=64$.

From which $x=2$; hence, $3x=6$ and $5x=10$, are the numbers. See general directions, page 127.

REVIEW.—211. In the solution of a problem producing an equation containing the second power of the unknown quantity, upon what principle is the equation found?

7. What two numbers are those which are to each other as 3 to 4, and the difference of whose squares is 63? Ans. 9 and 12.

8. What two numbers are those, which are to each other as 3 to 4, and the sum of whose squares is 100? Ans. 6 and 8.

9. What number is that, to which if 3 be added, and from which if 3 be subtracted, the product of the sum and difference is 40?

Ans. 7.

10. The breadth of a lot of ground is to its length, as 5 to 9, and it contains 1620 square feet; required the breadth and length.

Ans. Breadth 30, length 54 feet.

11. A man purchased a farm, giving $\frac{1}{10}$ as many dollars per acre, as there were acres in the farm; the cost of the farm was 1000 dollars; required the number of acres and the price per acre. Ans. 100 acres, $10 per acre.

12. What two numbers are those, whose sum is to the greater, as 10 to 7, and whose sum, multiplied by the less, produces 270?

Ans. 21 and 9.

Let $10x=$ their sum; then $7x=$ the greater, and $3x=$ the less number.

13. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Ans. 18 and 14.

14. C bought a number of oranges for 48 cents, and the price of an orange was to the number bought, as 1 to 3; how many did he buy, and how much a piece did he pay?

Ans. 12 oranges, at 4 cents a piece.

15. A person bought a piece of muslin for 3 dollars and 24 cents, and the number of cents which he paid for a yard, was to the number of yards, as 4 to 9; how many yards did he buy, and what was the price per yard? Ans. 27 yds., at 12 cents per yd

16. Find two numbers, in the ratio of $\frac{1}{2}$ to $\frac{2}{3}$, the sum of whose squares is 225. Ans. 9 and 12.

By reducing $\frac{1}{2}$ and $\frac{2}{3}$ to a common denominator, we find they are to each other as 3 to 4. Then let $3x$ and $4x$ represent the numbers.

17. Find three numbers, in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, the sum of whose squares is 724. Ans. 12, 16, and 18.

18. A merchant sold a piece of muslin at such a rate, that the price of a yard was to the number of yards, as 4 to 5; but, if he had received 45 cents more for the same piece, the price of a yard would have been to the number of yards as 5 to 4; how many yards were there in the piece, and what was the price per yard?

Ans. 10 yards, at 8 cents per yard.

COMPLETE EQUATIONS OF THE SECOND DEGREE.

1. Let it be required to find the values of x, in the equation

$$x^2-4x+4=1.$$

It is evident, from Article 197, that the first member of this equation is a perfect square. By extracting the square root of both members, we have $x-2=\pm 1$

Whence $x=2\pm 1=2+1=3$, or $2-1=1$.

Verification. $(3)^2-4\times 3+4=1$, that is, $9-12+4=1$

also, $(1)^2-4\times 1+4=1$, that is, $1-4+4=1$.

Hence, x has *two* values, $+3$ and $+1$, either of which verifies the equation.

2. Let it be required to find the value of x, in the equation

$$x^2+6x=16.$$

If the left member of this equation were a perfect square, we might find the value of x, by extracting the square root, as in the preceding example. To ascertain what is necessary to be added, to render the first member a perfect square, let us compare it with the square of $x+a$, which is $x^2+2ax+a^2$.

We find

$$\begin{aligned} x^2&=x^2 \\ 2ax&=6x \\ 2a&=6 \\ a&=3 \\ a^2&=9. \end{aligned}$$

Hence, by adding 9, *which is the square of half the coëfficient of the first power of* x, to each member, the equation becomes

$$x^2+6x+9=25$$

Extracting the square root, $x+3=\pm 5$

Whence $x=-3\pm 5=+2$, or -8.

Either of which values of x will verify the equation.

ART. **212.**—We will now show the different forms to which every complete equation of the second degree may be reduced, and illustrate further, the principle of completing the square.

Since every complete equation of the second degree may be reduced to the form $ax^2+bx=c$, if we divide both sides by a, we have

$$x^2+\frac{b}{a}x=\frac{c}{a}.$$

For the sake of simplicity, let $\frac{b}{a}=2p$, and $\frac{c}{a}=q$. The equation then becomes

$$x^2+2px=q \qquad (1.)$$

If $\frac{b}{a}$ is negative, and $\frac{c}{a}$ positive, the equation becomes

$$x^2-2px=q \qquad (2.)$$

If $\frac{b}{a}$ is positive, and $\frac{c}{a}$ negative, the equation becomes

$$x^2+2px=-q \quad (3.)$$

Lastly, if $\frac{b}{a}$ and $\frac{c}{a}$ are both negative, the equation becomes

$$x^2-2px=-q \quad (4.)$$

Hence, *every complete equation of the second degree, may be reduced to the form $x^2+2px=q$, in which $2p$ and q may be either positive or negative, integral or fractional quantities.*

We will now proceed to explain the principle, by which the first member of this equation may always be made a perfect square.

Since the square of a binomial is equal to the square of the first term, plus twice the product of the first term by the second, plus the square of the second; if we consider x^2+2px as the first two terms of the square of a binomial, x^2 is the square of the first term (x), and $2px$, the double product of the first term by the second; therefore, if we divide $2px$ by $2x$ (the double of the first term), or $2p$ by 2, the quotient, p (*half the coëfficient of x*), will be the second term of the binomial, and its square, p^2, added to the first member, will render it a perfect square. But, to preserve the equality, we must add the same quantity to both sides. This gives

$$x^2+2px+p^2=q+p^2$$

Extracting the square root, $\quad x+p=\pm\sqrt{q+p^2}$

Transposing, $\quad x=-p\pm\sqrt{q+p^2}.$

It is obvious, that the square may be completed in each of the other forms, on the same principle; that is, by taking half the coëfficient of the first power of x, squaring it, and adding it to each member. Thus, in the second form

$$x^2-2px=q$$
$$x^2-2px+p^2=q+p^2$$
$$x-p=\pm\sqrt{q+p^2}$$
$$x=+p\pm\sqrt{q+p^2}.$$

In the third and fourth forms, the values of x are readily obtained, in the same manner.

Collecting the four different forms together, and the values of x in each, we have the following table.

(1.)	$x^2+2px=q$	$x=-p\pm\sqrt{q+p^2}.$
(2.)	$x^2-2px=q.$	$x=+p\pm\sqrt{q+p^2}.$
(3.)	$x^2+2px=-q.$	$x=-p\pm\sqrt{-q+p^2}.$
(4.)	$x^2-2px=-q.$	$x=+p\pm\sqrt{-q+p^2}.$

Although the method of finding the value of x is the same in each of these forms, it is convenient to distinguish between them. See Art. 215.

From the preceding we derive the

RULE,

FOR THE SOLUTION OF A COMPLETE EQUATION OF THE SECOND DEGREE.

1st. *Reduce the equation, by clearing of fractions and transposition (if necessary), to the form* $ax^2+bx=c$.

2d. *Divide each side of the equation by the coëfficient of* x^2, *and add to each member the square of half the coëfficient of the first power of* x.

3d. *Extract the square root of both sides, and transpose the known term to the second member.*

EXAMPLES.

1. Find the roots of the equation $x^2+8x=33$.

Completing the square by taking half the coëfficient of $x(\frac{8}{2})$, squaring it, and adding the square to each member, we have

$$x^2+8x+16=33+16=49$$

Extracting the root, $x+4=\pm 7$

Transposing, $x=-4\pm 7$

Whence $x=-4+7=+3$

And $x=-4-7=-11$.

Verification. $(3)^2+8(3)=33$, that is, $9+24=33$.

Or $(-11)^2+8(-11)=33$, that is, $121-88=33$.

In verifying these values of x, it is to be noticed, that the square of -11, is 121, and that 8 multiplied by -11, gives -88.

2. Solve the equation $x^2-6x=16$.

Completing the square,

$$x^2-6x+9=16+9=25$$

Extracting the root, $x-3=\pm 5$

Transposing, $x=+3\pm 5$

Whence $x=+3+5=+8$

And $x=+3-5=-2$.

Both of which will be found to verify the equation.

3. Solve the equation $x^2+6x=-5$.

Completing the square,

$$x^2+6x+9=9-5=4$$

Extracting the root, $x+3=\pm 2$

Transposing, $x=-3\pm 2$

Whence $x=-3+2=-1$

And $x=-3-2=-5$.

4. Find the values of x, in the equation $x^2-10x=-24$.

Completing the square,

$$x^2-10x+25=25-24=1$$

Extracting the root, $x-5=\pm 1$

Transposing, $x=5\pm 1$

Whence $x=5+1=6$

And $x=5-1=4$.

The preceding examples, illustrate the four different forms, when the equation is already reduced. Equations of the second degree, however, generally occur in a more complicated form, and require to be reduced before completing the square.

5. Find the values of x, in the equation $3x-5=\dfrac{7x+36}{x}$.

Clearing of fractions, $3x^2-5x=7x+36$

Transposing, $3x^2-12x=36$

Dividing, $x^2-4x=12$

Completing the square,

$$x^2-4x+4=16$$

Extracting the root, $x-2=\pm 4$

Transposing, $x=+2\pm 4$

Whence $x=6$, or -2.

6. Find the values of x, in the equation $\dfrac{12x^2}{5}+x=52+\dfrac{13x}{5}$.

Clearing of fractions, $12x^2+5x=260+13x$

Transposing and reducing,

$$12x^2-8x=260$$

Dividing, $x^2-\frac{2}{3}x=\frac{65}{3}$.

Here the coëfficient of x is $-\frac{2}{3}$, the half of which is $-\frac{1}{3}$; the square of this is $\frac{1}{9}$, which being added to both sides, we have

$$x^2-\tfrac{2}{3}x+\tfrac{1}{9}=\tfrac{65}{3}+\tfrac{1}{9}=\tfrac{196}{9}$$

Extracting the root, $x-\frac{1}{3}=\pm\frac{14}{3}$

$x=+\frac{1}{3}\pm\frac{14}{3}$

Whence $x=+5$, or $-\frac{13}{3}$.

EXAMPLES FOR PRACTICE.

NOTE.—The first sixteen of the following Examples, are arranged to illustrate the four forms, to one of which every complete equation of the second degree may be reduced.

7. $x^2+8x=20$. . . . . Ans. $x=2$, or -10.
8. $x^2+16x=80$. . . . . Ans. $x=4$, or -20.
9. $x^2+7x=78$. . . . . Ans. $x=6$, or -13.
10. $x^2+3x=28$. . . . . Ans. $x=4$, or -7.

11. $x^2-10x=24$. . . . . Ans. $x=12$, or -2.
12. $x^2-8x=20$. . . . . Ans. $x=10$, or -2.
13. $x^2-5x=6$. . . . . Ans. $x=6$, or -1.
14. $x^2-21x=100$. . . . . Ans. $x=25$, or -4.

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15. $x^2+6x=-8$. . . . . Ans. $x=-2$, or -4.
16. $x^2+4x=-3$. . . . . Ans. $x=-1$, or -3.
17. $x^2+8x=-15$. . . . . Ans. $x=-3$, or -5.
18. $x^2+7x=-12$. . . . . Ans. $x=-3$, or -4.

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19. $x^2-6x=-8$. . . . . Ans. $x=4$, or 2.
20. $x^2-8x=-15$. . . . . Ans. $x=5$, or 3.
21. $x^2-10x=-21$. . . . . Ans. $x=7$, or 3.
22. $x^2-15x=-54$. . . . . Ans. $x=9$, or 6.

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23. $3x^2-2x+123=256$. . . . . Ans. $x=7$, or $-\frac{19}{3}$.
24. $2x^2-5x=12$. . . . . Ans. $x=4$, or $-\frac{3}{2}$.
25. $2x^2+3x=65$. . . . . Ans. $x=5$, or $-\frac{13}{2}$.
26. $\frac{2x^2}{3}-\frac{5x}{2}=\frac{2}{3}$. . . . . Ans. $x=4$, or $-\frac{1}{4}$.
27. $\frac{x^2}{100}=x-24$. . . . . Ans. $x=60$, or 40.
28. $x^2-x-40=170$. . . . . Ans. $x=15$, or -14.
29. $x=\frac{6-x}{x}$. . . . . Ans. $x=2$, or -3.
30. $x-1+\frac{2}{x-4}=0$. . . . . Ans. $x=3$, or 2.
31. $\frac{x}{4}-\frac{44}{x-2}=4$. . . . . Ans. $x=24$, or -6.
32. $\frac{5}{6}x^2-\frac{1}{2}x+\frac{3}{4}=8-\frac{2}{3}x-x^2+\frac{273}{12}$. . . . Ans. $x=4$, or $-\frac{45}{11}$.
33. $9x+\frac{1}{x}=\frac{29}{x}+4$. . . . . Ans. $x=2$, or $-\frac{14}{9}$.
34. $x^2+x=30$. . . . . Ans. $x=5$, or -6.
35. $\frac{x}{2}+\frac{2}{x}=\frac{x}{4}+\frac{3}{2}$. . . . . Ans. $x=2$, or 4.
36. $2x^2+92=31x$. . . . . Ans. $x=4$, or $11\frac{1}{2}$.
37. $-x^2+x=\frac{6}{25}$. . . . . Ans. $x=\frac{3}{5}$, or $\frac{3}{5}$.
38. $17x^2-19x=30$. . . . . Ans. $x=2$, or $-\frac{15}{17}$.

REVIEW.—212. To what form may every complete equation of the second degree be reduced? What are the four forms that this gives, depending on the signs of $2p$ and q? Explain the principle, by means of which the first member of the equation $x^2+2px=q$ may be made a perfect square. What is the rule for the solution of a complete equation of the second degree?

39. $3x^2+5x=2$. . . . . Ans. $x=\frac{1}{3}$, or -2.
40. $4x-3x^2=6x-8$. . . . . Ans. $x=\frac{4}{3}$, or -2.
41. $x^2-4x=-1$. . . . . Ans. $x=2\pm\sqrt{3}=3.732+$, or $.268-$.
42. $\frac{4x}{7}-\frac{2x^2}{3}=\frac{10x}{3}-\frac{20}{7}$. . . . . Ans. $x=-5$, or $\frac{6}{7}$.
43. $\frac{65x}{2}-\frac{10x^2}{11}=\frac{13}{2}-\frac{2x}{11}$. . . . . Ans. $x=35\frac{3}{4}$, or $\frac{1}{5}$.
44. $\frac{x}{x+8}=\frac{x+3}{2x+1}$. . . . . Ans. $x=12$, or -2.
45. $\frac{x}{x+60}=\frac{7}{3x-5}$. . . . . Ans. $x=14$, or -10.
46. $x+\frac{24}{x-1}=3x-4$. . . . . Ans. $x=5$, or -2.
47. $\frac{22-x}{20}=\frac{15-x}{x-6}$. . . . . Ans. $x=36$, or 12.
48. $\frac{x+3}{x}+\frac{7x}{x+3}=\frac{23}{4}$. . . . . Ans. $x=4$, or 1.
49. $\frac{2x+3}{10-x}=\frac{2x}{25-3x}-6\frac{1}{2}$. . . . . Ans. $x=8$, or $13\frac{22}{31}$.

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50. $2ax-x^2=-2ab-b^2$ . . . . Ans. $x=2a+b$, or $-b$.
51. $x^2-2ax=b^2-a^2$. . . . . Ans. $x=a+b$, or $a-b$.
52. $x^2+3bx-4b^2=0$. . . . . Ans. $x=+b$, or $-4b$.
53. $x^2-ax-bx=-ab$. . . . . Ans. $x=+a$, or $+b$.
54. $\frac{x}{x+a}=\frac{b}{x-b}$. . . . . Ans. $x=b\pm\sqrt{ab+b^2}$.
55. $2bx^2+(a-2b)x=a$. . . . . Ans. $x=1$, or $-\frac{a}{2b}$.
56. $\frac{x^2}{a^2}-\frac{x}{b}=\frac{2a^2}{b^2}$. . . . . Ans. $x=\frac{2a^2}{b}$ and $-\frac{a^2}{b}$.
57. $x^2-(a-1)x-a=0$. . . . . Ans. $x=a$, or -1.
58. $x^2-(a+b-c)x=(a+b)c$. . . . . . . Ans. $x=a+b$, or $-c$.

ART. **213.**—THE HINDOO METHOD OF SOLVING QUADRATICS.—When an equation is brought to the form $ax^2+bx=c$, it may be reduced to a simple equation, without dividing by the coëfficient of x^2; thus avoiding fractions.

If we multiply both sides of the equation $ax^2+bx=c$, by a, the coëfficient of x^2, it becomes $a^2x^2+abx=ac$.

Now, if we regard a^2x^2+abx, as the first and second terms of the square of a binomial, a^2x^2 must be the square of the first term, and abx the double product of the first term by the second. Hence, the first term of the binomial is $\sqrt{a^2x^2}=ax$; and the second term, the quotient derived from dividing abx by the double of ax, the

first term; that is, $\frac{abx}{2ax}=\frac{b}{2}$. Adding the square of $\frac{b}{2}$ to each side, the equation becomes $a^2x^2+abx+\frac{b^2}{4}=ac+\frac{b^2}{4}$.

Now, the left side is a perfect square; but it will still be a perfect square, if we multiply both sides by 4, which will clear it of fractions. Thus, $4a^2x^2+4abx+b^2=4ac+b^2$

Extracting the square root,

$$2ax+b=\pm\sqrt{4ac+b^2}$$

Whence $$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}.$$

Now, it is evident, that the equation $4a^2x^2+4abx+b^2=4ac+b^2$, may be derived directly from the equation $ax^2+bx=c$, by multiplying both sides by $4a$, the coëfficient of x^2, and then adding to each member, the square of b, the coëfficient of the first power of x. This gives the following

RULE,

FOR THE SOLUTION OF A COMPLETE EQUATION OF THE SECOND DEGREE.

Reduce the equation to the form $ax^2+bx=c$, and multiply both sides, by four times the coëfficient of x^2. Add the square of the coëfficient of x to each side, and then extract the square root. This will give a simple equation, from which x is easily found.

1. Given $3x^2-5x=28$, to find the values of x.

Multiplying both sides by 12, which is 4 times the coëfficient of x^2.

$$36x^2-60x=336$$

Adding to each member 25, the square of 5, the coëfficient of x,

$$36x^2-60x+25=361$$

Extracting the root, $6x-5=\pm19$

$$6x=5\pm19=24, \text{ or } -14$$

$$x=+4, \text{ or } -\tfrac{7}{3}.$$

By the same rule, find the values of the unknown quantity in each of the following examples.

2. $2x^2+5x=33$. . . . . Ans. $x=3$, or $-\frac{11}{2}$.
3. $5x^2+2x=88$. . . . . Ans. $x=4$, or $-\frac{22}{5}$.
4. $3x^2-x=70$. . . . . Ans. $x=5$, or $-\frac{14}{3}$.
5. $x^2-x=42$. . . . . Ans. $x=7$, or -6.
6. $\frac{1}{3}x^2+\frac{3x}{8}-5=9\frac{1}{4}$. . . . . Ans. $x=6$, or $-7\frac{1}{8}$.

If further exercises are desired, the examples in the preceding article may be solved by this rule.

REVIEW.—213. Explain the Hindoo method of completing the square.

PROBLEMS PRODUCING COMPLETE EQUATIONS OF THE SECOND DEGREE.

ART. **214.**—1. What number is that, whose square, diminished by the number itself, is equal to 20?

Let $x=$ the number.

Then $x^2-x=20$

Completing the square, $x^2-x+\frac{1}{4}=20+\frac{1}{4}=\frac{81}{4}$

Extracting the root, $x-\frac{1}{2}=\pm\frac{9}{2}$

Whence $x=+5$, or -4.

Now either of these values of x satisfies the equation; but the negative value -4, does not fulfill the conditions of the question in an arithmetical sense. But, since the subtraction of a negative quantity is equal to the addition of a positive quantity, the question may be so modified, that the value -4, will be a correct answer to it, the 4 being considered positive. The question thus changed, is: What number is that, whose square *increased* by the number itself, is equal to 20?

2. A person buys several oranges for 60 cents; had he bought 3 more for the same sum, each orange would have cost him 1 cent less; how many did he buy?

Let $x=$ the number he bought.

Then $\frac{60}{x}=$ the price of each one.

And $\frac{60}{x+3}=$ the price of one, had he bought 3 more for 60 cents.

Therefore, $\frac{60}{x}-\frac{60}{x+3}=1$

Clearing of fractions, and reducing,

$$x^2+3x=180$$

Completing the square, $x^2+3x+\frac{9}{4}=\frac{9}{4}+180=\frac{729}{4}$.

Extracting the root, $x+\frac{3}{2}=\pm\frac{27}{2}$

Whence $x=+12$, or -15.

Now either of these values, taken with its proper sign, satisfies the equation from which it was derived; but the value 12 is the only one that satisfies the conditions of the question.

Since $\frac{60}{-15}=-4$ and $\frac{60}{-15+3}=-5$; and since buying and selling are opposite operations, the result, -15, is the answer to this question. A person *sells* several oranges for 60 cents. Had he sold 3 *less* for the same sum, he would have *received* one cent *more* for each. How many oranges did he sell?

REMARK.—From the two preceding examples, we see, that the root which is obtained, from giving the plus sign to the radical, satisfies both

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the conditions of the question, and the equation derived from it; while the other root satisfies the equation only.

We see, also, that the root which arises from giving the radical the negative sign, may be regarded as the answer to a question differing from the one proposed in this; that certain quantities which were *additive*, have become *subtractive*, and reciprocally.

Sometimes, however, as in the following example, both values of the unknown quantity satisfy the conditions of the question.

3. Find a number, whose square increased by 15, shall be 8 times the number.

Let $x=$ the number; then $x^2+15=8x$

Or $x^2-8x=-15$

Whence $x=5$, or 3.

Either of which fulfills the conditions of the question.

When there are two unknown quantities in a problem, that can be solved by the use of one symbol, the two values of the symbol generally give both values of the unknown quantity, as in the following question.

4. Divide the number 24 into two such parts, that their product shall be 95.

Let $x=$ one of the parts; then $24-x=$ the other.

And $x(24-x)=95$

Or $x^2-24x=-95$

Whence $x=19$ and 5

And $24-x=5$, or 19.

5. There are three numbers, such that the product of the first and third is equal to the square of the second; the sum of the first and second is 10, and the third exceeds the second, by 24; required the numbers.

Let $x=$ the first; then $10-x=$ the second,

And $10-x+24=34-x=$ the third.

Also $(10-x)^2=x(34-x)$

Or $100-20x+x^2=34x-x^2$

From which, $x=25$, or 2.

When $x=25$, $10-x=-15$, $34-x=9$, and the numbers are 25, -15, and 9.

When $x=2$, $10-x=8$, $34-x=32$, and the numbers are 2, 8, and 32.

Both these sets of values satisfy the question in an algebraic sense; only the last, however, satisfies it in an arithmetical sense. Let us endeavor to ascertain how the question must be modified, so that the first set of numbers shall satisfy it in an arithmetical sense.

The meaning of the negative solution -15, will be understood by considering that the addition of a negative quantity, is the same as the subtraction of the same quantity taken positively (Art 61). The first condition of the question then becomes $25+(-15)=25-(+15)=25-15=10$; and the second is $9-(-15)=9+(+15)=9+15=24$. This indicates, that -15 may be changed to $+15$, provided, that instead of the condition of the *sum* of the first and second numbers being 10, their *difference* be 10; and the second condition may for a similar reason, be changed into this, that the *sum* of the second and third is 24. The question, with these modifications, would be: What three numbers are those, such that the product of the first and third, is equal to the square of the second; the *difference* of the first and second is 10; and the *sum* of the second and third is 24?

REMARK.—In the following examples, the pupil is required to find only that value of the unknown quantity, which satisfies the conditions of the question in an arithmetical sense. It forms, however, a good exercise for advanced pupils, to determine the negative value, and then to modify the question, so that this value shall satisfy the conditions in an arithmetical sense.

6. Find a number, such that if its square be diminished by 6 times the number itself, the remainder shall be 7. Ans. 7.

7. Find a number, such that if its square be increased by 8 times the number itself, the sum shall be 9. Ans. 1.

8. Find a number, such that twice its square, plus 3 times the number itself, shall be 65. Ans. 5.

9. Find a number, such that if its square be diminished by 1, and $\frac{2}{3}$ of the remainder be taken, the result shall be equal to 5 times the number divided by 2. Ans. 4.

10. Find a number, such that if 44 be divided by the number diminished by 2, the quotient shall be equal to $\frac{1}{4}$ of the number, diminished by 4. Ans. 24.

11. Find two numbers, whose difference is 8, and product 240. Ans. 12 and 20.

12. A person bought a number of sheep, for 80 dollars; if he had bought 4 more for the same money, he would have paid 1 dollar less for each; how many did he buy? Ans. 16.

13. There are two numbers, whose difference is 10, and if 600 be divided by each, the difference of the quotients is also 10; what are the numbers? Ans. 20 and 30.

14. A pedestrian, having to walk 45 miles, finds that if he increases his speed $\frac{1}{2}$ a mile an hour, he will perform his task $1\frac{1}{4}$

hours sooner, than if he walked at his usual rate; what is that rate? Ans. 4 miles per hour.

15. Divide the number 14 into two parts, the sum of whose squares shall be 100. Ans. 8 and 6.

16. In an orchard containing 204 trees, there are 5 more trees in a row than there are rows; required the number of rows, and the number of trees in a row. A. 12 rows, and 17 trees in a row.

17. A schoolboy, being asked the ages of his sister and himself, replied, that he was 4 years older than his sister, and that twice the square of her age, was 7 less than the square of his own; required their ages. Ans. 13 and 9 yrs.

18. A and B start at the same time to travel 150 miles; A travels 3 miles an hour faster than B, and finishes his journey $8\frac{1}{3}$ hours before him; at what rate per hour did each travel?

Ans. 9 and 6 miles per hour.

19. A company at a tavern had 1 dollar and 75 cents to pay; but before the bill was paid two of them went away, when those who remained had each 10 cents more to pay; how many were in the company at first? Ans. 7

20. The product of two numbers is 100, and if 1 be taken from the greater, and added to the less, the product of the resulting numbers is 120; what are the numbers? Ans. 25 and 4.

Let $x=$ the larger number; then $\frac{100}{x}=$ the smaller.

21. If 4 be subtracted from a father's age, the remainder will be thrice the age of the son; and if 1 be taken from the son's age, half the remainder will be the square root of the father's age. Required the age of each. Ans. 49 and 15 yrs.

Let $x^2=$ the father's age; then $\frac{x^2-4}{3}=$ the son's age.

22. A young lady being asked her age, answered, "If you add the square root of my age to $\frac{3}{8}$ of my age, the sum will be 10." Required her age. Ans. 16 yrs.

23. What number is that, from which, if $\frac{3}{5}$ of its square root be taken, the remainder will be 22? Ans. 25.

24. A merchant bought a piece of muslin for 6 dollars; after cutting off 15 yards, he sold the remainder for 5 dollars 40 cents, by which he gained 1 cent a yard on the amount sold; how many yards did he buy, and at what price?

Ans. 75 yds., at 8 cts. per yd.

25. A man bought a horse, which he afterward sold for 24 dollars, and by so doing, lost as much per cent. upon the price of his purchase, as the horse cost him; what did he pay for the horse?

Ans. \$60, or \$40.

PROPERTIES OF THE ROOTS OF A COMPLETE EQUATION OF THE SECOND DEGREE.

NOTE TO TEACHERS.—This subject may be omitted entirely, by the younger class of pupils; and passed over, by those more advanced, until the book is reviewed.

ART. **215.**—The pupil may have learned already, by inference, from the solution of the preceding examples, that an equation of the second degree has two roots, that is, that the unknown quantity has two values. This principle may be proved directly, as follows.

The general form to which every complete equation of the second degree may be reduced, is $x^2+2px=q$; in which $2p$ and q may be either both positive or both negative, or one positive and the other negative. Completing the square, we have

$$x^2+2px+p^2=q+p^2$$

Now, the first member is equal to $(x+p)^2$, and if, for the sake of simplicity, we assume $q+p^2=m^2$, that is, $\sqrt{q+p^2}=m$, then

$$(x+p)^2=m^2$$

Transposing, $\quad (x+p)^2-m^2=0.$

But, since the left hand member of this equation, is the difference of two squares, it may be resolved into two factors, Art. 94. This gives $\quad (x+p+m)(x+p-m)=0$

Now, this equation can be satisfied in two ways, and in *only* two; that is, by making either of the factors equal to 0.

If we make the second factor equal to zero, we have

$$x+p-m=0$$

Or, by transposing, $\quad x=-p+m=-p+\sqrt{q+p^2}.$

If we make the first factor equal to zero, we have

$$x+p+m=0$$

Or, by transposing, $\quad x=-p-m=-p-\sqrt{q+p^2}.$

Hence, we conclude,

1st. *That every equation of the second degree, has two roots, and only two.*

2d. *That every complete equation of the second degree, reduced to the form $x^2+2px=q$, may be decomposed into two binomial factors, of which the first term in each is x, and the second, the two roots with their signs changed.*

Thus, the two roots of the equation $x^2-5x=-6$, or $x^2-5x+6=0$, are $x=2$ and $x=3$; hence, $x^2-5x+6=(x-2)(x-3)$.

From this, it is evident, that the direct method of resolving a quadratic trinomial into its factors, is to place it equal to zero, and then find the roots of the equation. In this manner, let the learner solve the questions on page 72.

By reversing the operation, we can readily form an equation, whose roots shall have any given values.

Thus, let it be required to form an equation whose roots shall be 4 and -6.

We must have $x=4$ or $x-4=0$

And $x=-6$ or $x+6=0$

Hence, $(x-4)(x+6)=x^2+2x-24=0$

Or $x^2+2x=24$.

Which is an equation whose roots are $+4$ and -6.

1. Find an equation whose roots are 7 and 10.

Ans. $x^2-17x=-70$.

2. Find an equation whose roots are -3 and -1.

Ans. $x^2+4x=-3$.

3. Find an equation whose roots are $+2$, and -1.

Ans. $x^2-x=2$.

ART. **216.**—Resuming the equation $x^2+2px=q$.

The first value of x is $-p+\sqrt{q+p^2}$

The second value of x is $-p-\sqrt{q+p^2}$

Their sum is $-2p$, which is the coëfficient of x, taken with a contrary sign. Hence, we conclude,

That the sum of the roots of an equation of the second degree, reduced to the form $x^2+2px=q$, is equal to the coëfficient of the first power of x taken with a contrary sign.

If we take the product of the roots, we have

First root$=$ $-p+\sqrt{q+p^2}$

Second root$=$ $-p-\sqrt{q+p^2}$

$$\begin{array}{l} p^2-p\sqrt{q+p^2} \\ \quad +p\sqrt{q+p^2} \quad (q+p^2) \\ \hline p^2 \quad . \quad . \quad . \quad . \quad -(q+p^2)=-q. \end{array}$$

But $-q$ is the known term of the equation, taken with a contrary sign. Hence, we conclude,

That the product of the two roots of an equation of the second degree, reduced to the form $x^2+2px=q$, is equal to the known term taken with a contrary sign.

REMARK.—In the preceding demonstrations, we have regarded $2p$ and q as both positive; the same course of reasoning, however, will apply when they are both negative, or when one is positive and the other negative; so that the conclusions are true in each of the four different forms.

ART. **217.**—In the equation $x^2+2px=q$, or first form,

the two values of x are $-p+\sqrt{q+p^2}$

And $-p-\sqrt{q+p^2}$.

If we examine the part $\sqrt{q+p^2}$, we see that its value must be a quantity greater than p, since the square root of p^2 alone, is p. Hence, the first root is the difference between p and a positive quantity greater than p; therefore, it is *essentially positive.*

If we take the negative value of the radical part, the second root is equal to the sum of two negative quantities, one of which is p, and the other a quantity greater than p; therefore, it is *essentially negative.* Since the first root is the *difference*, and the second root the *sum*, of the same two quantities, the second, or negative root, is necessarily greater than the first, or positive root See questions 7, 8, 9, 10, page 205.

In the equation $x^2-2px=q$, or second form,

the two values of x are $\quad +p+\sqrt{q+p^2}$

And $\quad +p-\sqrt{q+p^2}$.

The quantity under the radical being the same as in the preceding form, its square root is greater than p. The first root then, is the *sum* of two positive quantities, one of which is p, and the other a quantity greater than p; therefore, it is *essentially positive.*

If we take the negative value of the radical part, the second root is equal to the difference between p, and a negative quantity greater than p; therefore it is *essentially negative.*

Since the first root is the *sum*, and the second root the *difference* of the same two quantities, the first, or positive root, is greater than the second, or negative root. See questions 11, 12, 13, 14, page 206.

In the equation $x^2+2px=-q$, or third form,

the two values of x are $\quad -p+\sqrt{-q+p^2}$

And $\quad -p-\sqrt{-q+p^2}$.

If we examine the radical part, $\sqrt{-q+p^2}$, we see, that its value must be a quantity less than p, since the square root of p^2 without its being diminished, is p; hence, the first root is the difference between $-p$, and a positive quantity less than p; therefore, it is *essentially negative.*

If we take the negative value of the radical part, the second root is equal to the sum of two negative quantities; therefore, it is *essentially negative.*

REVIEW.—215. To what general form, may every equation of the second degree, containing one unknown quantity, be reduced? Show that every equation of the second degree has two roots, and only two. 216. To what is the sum of the roots of an equation of the second degree equal? To what is the product equal? 217. Show that in the first form one of the roots is positive, and the other negative; and that the negative root is greater than the positive.

Hence, in the third form, both roots are negative. See questions 15, 16, 17, 18, page 206.

In the equation $x^2-2px=-q$, or fourth form, the two values of x are $+p+\sqrt{-q+p^2}$
And $+p-\sqrt{-q+p^2}$.

The value of the radical part, being the same as in the preceding form, it is less than p. The first root, then, is the *sum* of two positive quantities, therefore, it is *essentially positive.*

The second root is the difference between p, and a negative quantity less than p, therefore, it is *essentially positive.*

Hence, in the fourth form, both roots are positive. See questions 19, 20, 21, 22, page 206.

ART. **218.**—In the third and fourth forms, the radical part is $\sqrt{-q+p^2}$. Now, if q is greater than p^2, this is essentially negative, and we are required to extract the square root of a negative quantity, which is impossible. See Art. 195. Therefore, in the third and fourth forms, when q is greater than p^2, that is, when the known term is negative, and greater than the square of half the coëfficient of the first power of x, both values of the unknown quantity are impossible. What is the cause of this impossibility?

To explain this, we must inquire into what two parts, a number must be divided, so that the product of the parts shall be the greatest possible.

Let $2p$ represent any number, and let the parts, into which it is supposed to be divided, be $p+z$ and $p-z$. The product of these parts is $(p+z)(p-z)=p^2-z^2$.

Now, this product is evidently the greatest, when z^2 is the least; that is, when z^2 or z is 0. But, when z is 0, the parts are p and p, that is, *when a number is divided into two equal parts, their product is greater than that of any other two parts into which the number can be divided.* Or, as the same principle may be otherwise expressed, *the product of any two unequal numbers is less than the square of half their sum.*

As an illustration of this principle, take the number 10, and divide it into different parts.

$10=9+1$, and $9\times1=9$
$10=8+2$, and $8\times2=16$
$10=7+3$, and $7\times3=21$
$10=6+4$, and $6\times4=24$
$10=5+5$, and $5\times5=25$

REVIEW.—217. Show that in the second form, one root is positive and the other negative; and that the positive root is greater than the negative. Show that in the third form, both roots are negative. Show that in the fourth form, both roots are positive.

We thus see, that the product of the parts becomes greater as they approach to equality, and that it is the greatest when they are equal to each other.

Now, in Art. 216, it has been shown, that $2p$, the coëfficient of the first power of x, is equal to the sum of the two values of x, and that q is equal to their product. But, when q is greater than p^2, we have the product of two numbers, greater than the square of half their sum, which, by the preceding theorem, is impossible. If, then, any problem furnishes an equation in which the known term is negative, and greater than the square of half the coëfficient of the first power of the unknown quantity, we infer, that the conditions of the problem are incompatible with each other. The following is an example.

Let it be required to divide the number 12 into two such parts, that their product shall be 40.

Let x and $12-x$ represent the parts.

Then $$x(12-x)=40, \text{ or } x^2-12x=-40$$
$$x^2-12x+36=-4$$
$$x-6=\pm\sqrt{-4}, \text{ and } x=6\pm\sqrt{-4}.$$

These expressions for the values of x, show that the problem is impossible. This we also know, from the preceding theorem, since the number 12 can not be divided into any two parts, whose product will be greater than 36; thus, the algebraic solution renders manifest the absurdity of an impossible problem.

REMARKS.—1st. When the coëfficient of x^2 is negative, as in the equation $-x^2+mx=n$, the pupil may not perceive that it is embraced in the four general forms. This difficulty is obviated, by multiplying both sides of the equation by -1.

2d. Since the sign of the square root of x^2, or of $(x+p)^2$, is $\pm$, it might seem, that when $x^2=m^2$, we should have $\pm x=\pm m$, that is, $+x=\pm m$, and $-x=\pm m$; such is really the case, but $-x=+m$, is the same as $+x=-m$, and $-x=-m$, is the same as $+x=+m$. Hence, $+x=\pm m$, embraces all the values of x. In the same manner, it is necessary to take only the plus sign of the square root of $(x+p)^2$.

EQUATIONS OF THE SECOND DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

NOTE.—A full discussion of equations of this class does not properly belong to an elementary treatise. Indeed, no directions can be given, that will be applicable to all cases. The general method of treating the subject, consists in presenting the solution of a variety of examples, and then furnishing others for the exercise of the student. The following examples are intended to embrace only those capable of solution by simple methods. See Ray's Algebra, Part Second.

ART. **219.**—In solving equations of the second degree, containing two unknown quantities, the first step is to eliminate one of them, so as to obtain a single equation involving only one unknown quantity. The elimination may be performed by either of the three methods already given. See Articles 158, 159, 160. When a single equation is thus obtained, the value of the unknown quantity is to be found by the rules already given.

EXAMPLES.

1. Given $x-y=2$ and $x^2+y^2=100$, to find x and y.

By the first equation, $x=y+2$

Substituting this value of x, in the second,

$$(y+2)^2+y^2=100$$

From which we readily find, $y=6$, or -8

Hence, $x=y+2=8$, or -6.

2. Given $x+y=8$, and $xy=15$, to find x and y.

From the first equation, $x=8-y$

Substituting this value of x, in the second,

$$y(8-y)=15$$

Or $y^2-8y=-15$

From which y is found to be 5 or 3.

Hence, $x=3$, or 5.

There is a general method of solving questions of this form, without completing the square, with which pupils should be acquainted. To explain it, suppose we have the equations

$$x+y=a$$
$$xy=b$$

Squaring the first, $x^2+2xy+y^2=a^2$

Multiplying the second by 4, $4xy=4b$

Subtracting, $x^2-2xy+y^2=a^2-4b$

Extracting the square root, $x-y=\pm\sqrt{a^2-4b}$

But $x+y=a$

Adding $2x=a\pm\sqrt{a^2-4b}$

Or $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}$

Subtracting, $2y=a\mp\sqrt{a^2-4b}$

Or $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{a^2-4b}$.

If we have the equations $x-y=a$ and $xy=b$, we may find the values of x and y, in a similar manner, by squaring each member of the first equation, and *adding* to each side 4 times the second. Then, extracting the square root, we obtain the value of $x+y=\pm\sqrt{a^2+4b}$; from which, and $x-y=a$, we find $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2+4b}$, and $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{a^2+4b}$.

3. Given $x+y=a$ and $x^2+y^2=b$, to find x and y.

Squaring the first, $x^2+2xy+y^2=a^2$ (3.)

But, $x^2+y^2=b$ (2.)

Subtracting, $2xy=a^2-b$ (4.)

Take (4) from (2), $x^2-2xy+y^2=2b-a^2$

Extracting the root $x-y=\pm\sqrt{2b-a^2}$

$x+y=a$

Adding and dividing, $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{2b-a^2}$

Subtracting and dividing, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{2b-a^2}$.

4. Given $x^2+y^2=a$ and $xy=b$, to find x and y.

Adding twice the second to the first,

$$x^2+2xy+y^2=a+2b$$

Extracting the square root, $x+y=\pm\sqrt{a+2b}$

Subtracting twice the second from the first,

$$x^2-2xy+y^2=a-2b$$

Extracting the square root, $x-y=\pm\sqrt{a-2b}$

Whence $x=\pm\frac{1}{2}\sqrt{a+2b}\pm\frac{1}{2}\sqrt{a-2b}$

And $y=\pm\frac{1}{2}\sqrt{a+2b}\mp\frac{1}{2}\sqrt{a-2b}$.

5. Given $x^3+y^3=a$ and $x+y=b$, to find x and y.

Dividing the first by the second,

$$x^2-xy+y^2=\frac{a}{b} \quad (3.)$$

Squaring the second, $x^2+2xy+y^2=b^2$ (4.)

Subtracting (3) from (4), $3xy=\frac{b^3-a}{b}$

Or $xy=\frac{b^3-a}{3b}$ (5.)

Take (5) from (3), $x^2-2xy+y^2=\frac{4a-b^3}{3b}$

Extracting the root, $x-y=\pm\sqrt{\left(\frac{4a-b^3}{3b}\right)}$

But $x+y=b$

Whence $x=\frac{1}{2}b\pm\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}$

And $y=\frac{1}{2}b\mp\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}$

In a similar manner, if we have $x^3-y^3=a$ and $x-y=b$ we find $x=\pm\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}+\frac{1}{2}b$, and $y=\pm\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}-\frac{1}{2}b$.

EXAMPLES.

6. $\left.\begin{array}{l} x^2+y^2=34 \\ x^2-y^2=16 \end{array}\right\}$ . . . . Ans. $x=\pm 5$. $y=\pm 3$.

7. $\left.\begin{array}{l} x+y=16 \\ xy=63 \end{array}\right\}$ . . . . Ans. $x=9$, or 7. $y=7$, or 9.

8. $\left.\begin{array}{l} x-y=5 \\ xy=36 \end{array}\right\}$ . . . . Ans. $x=9$, or -4. $y=4$, or -9.

9. $\left.\begin{array}{l} x+y=9 \\ x^2+y^2=53 \end{array}\right\}$ . . . . Ans. $x=7$, or 2. $y=2$, or 7.

10. $\left.\begin{array}{l} x-y=5 \\ x^2+y^2=73 \end{array}\right\}$ . . . . Ans. $x=8$, or -3. $y=3$, or -8.

11. $\left.\begin{array}{l} x^3+y^3=152 \\ x+y=8 \end{array}\right\}$ . . . . Ans. $x=5$, or 3. $y=3$, or 5.

12. $\left.\begin{array}{l} x^3-y^3=208 \\ x-y=4 \end{array}\right\}$ . . . . Ans. $x=6$, or -2. $y=2$, or -6.

13. $\left.\begin{array}{l} x^3+y^3=19(x+y) \\ x-y=3 \end{array}\right\}$ . . . . Ans. $x=5$, or -2. $y=2$, or -5.

14 $\left.\begin{array}{l} x+y=11 \\ x^2-y^2=11 \end{array}\right\}$ . . . . Ans. $x=6$. $y=5$.

15. $\left.\begin{array}{r} (x-3)(y+2)=12 \\ xy=12 \end{array}\right\}$ . . . . Ans. $x=6$, or -3. $y=2$, or -4.

16. $\left.\begin{array}{l} y-x=2 \\ 3xy=10x+y \end{array}\right\}$ . . . . Ans. $x=2$, or $-\frac{1}{3}$. $y=4$, or $1\frac{2}{3}$.

17. $\left.\begin{array}{l} 3x^2+2xy=24 \\ 5x-3y=1 \end{array}\right\}$ . . . . Ans. $x=2$, or $-\frac{36}{19}$. $y=3$, or $-\frac{199}{57}$.

18. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{5}{6} \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{13}{36} \end{array}\right\}$ . . . . Ans. $x=2$, or 3. $y=3$, or 2.

19. $\left.\begin{array}{l} x-y=2 \\ x^2y^2=21-4xy \end{array}\right\}$ . . . . Ans. $x=3$, or -1. $y=1$, or -3.

In solving question 18, let $\frac{1}{x}=v$, and $\frac{1}{y}=z$; the question then becomes similar to the 9th. In question 19, find the value of xy from the second equation, as if it were a single unknown quantity.

PROBLEMS PRODUCING EQUATIONS OF THE SECOND DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

1. The sum of two numbers is 10, and the sum of their squares 52; what are the numbers? Ans. 4 and 6.

2. The difference of two numbers is 3, and the difference of their squares 39; required the numbers. Ans. 8 and 5.

3. It is required to divide the number 25 into two such parts, that the sum of their square roots shall be 7. Ans. 16 and 9.

4. The product of a certain number, consisting of two places, by the sum of its digits, is 160, and if it be divided by 4 times the digit in unit's place, the quotient is 4; required the number.
Ans. 32.

5. The difference between two numbers, multiplied by the greater, =16, but by the less, =12; required the numbers.
Ans. 8 and 6.

6. Divide 10 into two such parts, that their product shall exceed their difference by 22. Ans. 6 and 4.

7. The sum of two numbers is 10, and the sum of their cubes is 370; required the numbers. Ans. 3 and 7.

8. The difference of two numbers is 2, and the difference of their cubes is 98; required the numbers. Ans. 5 and 3.

9. The sum of 6 times the greater of two numbers, and 5 times the less, is 50, and their product is 20; required the numbers.
Ans. 5 and 4.

10. If a certain number, consisting of two places, is divided by the product of its digits, the quotient will be 2, and if 27 is added to it, the digits will be inverted; required the number.
Ans. 36.

11. Find three such quantities, that the quotients arising from dividing the products of every two of them, by the one remaining, are a, b, and c. Ans. $\pm\sqrt{ab}$, $\pm\sqrt{ac}$, and $\pm\sqrt{bc}$.

12. The sum of two numbers is 9, and the sum of their cubes is 21 times as great as their sum; required the numbers.
Ans. 4 and 5.

13. There are two numbers, the sum of whose squares exceeds twice their product, by 4, and the difference of their squares exceeds half their product, by 4; required the numbers.
Ans. 6 and 8.

14. The fore wheel of a carriage makes 6 revolutions more than the hind wheel, in going 120 yards; but if the circumference of each wheel is increased 1 yard, it will make only 4 revolutions more than the hind wheel, in the same distance; required the circumference of each wheel. Ans. 4 and 5 yds.

15. Two persons, A and B, depart from the same place, and travel in the same direction; A starts 2 hours before B, and after traveling 30 miles, B overtakes A; but had each of them traveled half a mile more per hour, B would have traveled 42 miles before overtaking A. At what rate did they travel?
Ans. A $2\frac{1}{2}$, and B 3 miles per hour.

16. A and B started at the same time, from two different points, toward each other; when they met on the road, it appeared that A had traveled 30 miles more than B. It also appeared, that it would take A 4 days to travel the road that B had come, and B 9 days to travel the road that A had come. Find the distance of A from B, when they set out. Ans. 150 miles.

CHAPTER VIII.

PROGRESSIONS AND PROPORTION.

ARITHMETICAL PROGRESSION.

ART. **220.**—A *series,* is a collection of quantities or numbers, connected together by the signs + or —, and in which any one term may be derived from those which precede it, by a rule, which is called the *law* of the series. Thus,

$$1+3+5+7+9+, \text{\&c.},$$
$$2+6+18+54+, \text{\&c.},$$

are series; in the former of which, any term may be derived from that which precedes it, by adding 2; and in the latter, any term may be found by multiplying the preceding term by 3.

ART. **221.**—An *Arithmetical Progression* is a series of quantities which increase or decrease, by a *common difference.* Thus, the numbers 1, 3, 5, 7, 9, &c., form an *increasing* arithmetical progression, in which the common difference is 2.

The numbers 30, 27, 24, 21, &c., form a *decreasing* arithmetical progression, in which the common difference is 3.

REMARK.—An arithmetical progression is termed, by some writers, an *equidifferent series,* or a *progression by differences.*

Again, a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c., is an *increasing* arithmetical progression, whose first term is a, and common difference d. And if d be negative, it becomes a, $a-d$, $a-2d$, $a-3d$, $a-4d$, &c., which is a *decreasing* arithmetical progression, whose first term is a, and common difference d.

ART. **222.**—If we take an arithmetical series, of which the first term is a and common difference d, we have

$$\begin{aligned}
&\text{1st term} = \ldots\ldots\ldots a\\
&\text{2d term} = \text{1st term} + d = a+d\\
&\text{3d term} = \text{2d term} + d = a+2d\\
&\text{4th term} = \text{3d term} + d = a+3d, \text{ and so on.}
\end{aligned}$$

Hence, the coëfficient of d in any term, is less by unity, than the number of that term in the series; therefore, the nth term $=a+(n-1)d$.

If we designate the nth term by l, we have $l=a+(n-1)d$.

Hence, the

RULE,

FOR FINDING ANY TERM OF AN INCREASING ARITHMETICAL SERIES.

Multiply the common difference by the number of terms less one, and add the product to the first term; the sum will be the required term.

If the series is decreasing, then d is *minus*, and the formula is $l=a-(n-1)d$. This gives the

RULE,

FOR FINDING ANY TERM OF A DECREASING ARITHMETICAL SERIES.

Multiply the common difference by the number of terms less one, and subtract the product from the first term; the remainder will be the required term.

EXAMPLES.

1. The first term of an increasing arithmetical series is 3, and common difference 5; required the 8th term.

Here l, or 8th term $=3+(8-1)5=3+35=38$. Ans.

2. The first term of a decreasing arithmetical series is 50, and common difference 3; required the 10th term.

Here l, or 10th term $=50-(10-1)3=50-27=23$. Ans.

In the following examples, a denotes the first term, and d the common difference of an arithmetical series; d being *plus* when the series is *increasing*, and *minus* when it is *decreasing*.

3. $a=3$, and $d=5$; required the 6th term. . . . . Ans. 28.
4. $a=20$, and $d=4$; required the 15th term. . . . Ans. 76.
5. $a=7$, and $d=\frac{1}{4}$; required the 16th term. . . . Ans. $10\frac{3}{4}$.
6. $a=2\frac{1}{2}$, and $d=\frac{1}{3}$; required the 100th term. . Ans. $35\frac{1}{2}$.
7. $a=0$, and $d=\frac{1}{2}$; required the 11th term. . . . . Ans. 5.
8. $a=30$, and $d=-2$; required the 8th term. . . . Ans. 16.
9. $a=-4$, and $d=3$; required the 5th term. . . . Ans. 8.
10. $a=-10$, and $d=-2$; required the 6th term. Ans. -20.
11. If a body falls during 20 seconds, descending $16\frac{1}{12}$ feet the first second, $48\frac{1}{4}$ feet the next, and so on, how far will it fall the twentieth second? Ans. $627\frac{1}{4}$ feet.

REVIEW.—220. What is a series? Give examples. 221. What is an arithmetical progression? Give an example of an increasing series. Of a decreasing series. 222. What is the rule for finding the last term of an increasing arithmetical series? Of a decreasing arithmetical series? Explain the reason of these rules.

ART. **223.**—Given, the first term a, the common difference d, and the number of terms n, to find s, the sum of the series.

If we take an arithmetical series of which the first term is **3**, common difference 2, and number of terms 5, it may be written in the following forms:

$$3,\quad 3+2,\quad 3+4,\quad 3+6,\quad 3+8$$
$$11,\quad 11-2,\quad 11-4,\quad 11-6,\quad 11-8.$$

It is obvious, that the sum of all the terms in either of these lines, will represent the sum of the series; that is,

$$s=3+(3+2)+(3+4)+(3+6)+(3+8)$$

And $\quad s=11+(11-2)+(11-4)+(11-6)+(11-8)$

Adding, $2s=14+14+14+14+14$

$=14\times$ the number of terms.

$=14\times5=70$

Whence, $\quad s=\frac{1}{2}$ of $70=35$.

Now, let $l=$ the last term, then writing the series both in a direct and inverted order,

$$s=a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots+l$$

And $\quad s=l+(l-d)+(l-2d)+(l-3d)+\ldots\ldots+a$

By adding the corresponding terms, we have

$$2s=(l+a)+(l+a)+(l+a)+(l+a)\ldots+(l+a)$$

$=(l+a)$ taken as many times as there are terms (n) in the series.

Hence, $\quad 2s=(l+a)n$

$$s=(l+a)\frac{n}{2}=\left(\frac{l+a}{2}\right)n.$$

This formula gives the following

RULE,

FOR FINDING THE SUM OF AN ARITHMETICAL SERIES.

Multiply half the sum of the two extremes, by the number of terms.

From the preceding, it appears, that *the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.*

REMARK.—Since $l=a+(n-1)d$, if we substitute this in the place of l in the formula $s=(l+a)\frac{n}{2}$, it becomes $s=\left(2a+(n-1)d\right)\frac{n}{2}$. This gives the following Rule, for finding the sum of an arithmetical series: *To the double of the first term add the product of the number of terms less one, by the common difference, and multiply the sum by half the number of terms.*

REVIEW.—223. What is the rule for finding the sum of an arithmetical series? Explain the reason of the rule.

EXAMPLES.

1. Find the sum of an arithmetical series, of which the first term is 3, last term 17, and number of terms 8.

$$s=\left(\frac{3+17}{2}\right)8=80. \quad \text{Ans.}$$

2. Find the sum of an arithmetical series, whose first term is 1, last term 12, and number of terms 12. Ans. 78.

3. Find the sum of an arithmetical series, whose first term is 0, common difference 1, and number of terms 20. Ans. 190.

4. Find the sum of an arithmetical series, whose first term is 3, common difference 2, and number of terms 21. Ans. 483.

5. Find the sum of an arithmetical series, whose first term is 10, common difference —3, and number of terms 10. A. —35.

In this case, the sum of the negative terms exceeds that of the positive.

ART. **224.**—The equations $l=a+(n-1)d$ and

$$s=(a+l)\frac{n}{2}, \text{ furnish the means of}$$

solving this general problem: *Knowing any three of the five quantities a, d, n, l, s, which enter into an arithmetical series, to determine the other two.*

This question furnishes ten problems, the solution of which pre sents no difficulty; for we have always two equations, to determine the two unknown quantities, and the equations to be solved, are either those of the first or second degree.

1. Let it be required to find a in terms of l, n, and d.

From the first formula, by transposing, we have $a=l-(n-1)d$. That is, *the first term of an increasing arithmetical series is equal to the last term diminished by the product of the common difference into the number of terms less one.*

From the same formula, by transposing a, and dividing by $n-1$, we find $d=\frac{l-a}{n-1}$.

That is, *in any arithmetical series, the common difference is equal to the difference of the extremes, divided by the number of terms less one.*

Examples, illustrating these principles, will be found in the collection at the close of this subject.

REVIEW.—224. What are the fundamental equations of arithmetical progression, and to what general problem do they give rise? To what is the first term of an increasing arithmetical series equal? To what is the common difference of an arithmetical series equal?

ART. **225.**—By means of the preceding principle, we are enabled to solve the following problem.

Two numbers, a and b, being given, to insert a number, m, of arithmetical means between them; that is, so that the numbers inserted, shall form, with the two given numbers, an arithmetical series.

Regarding a and b as the first and last terms of an increasing arithmetical series, if we insert m terms between them, we shall have a series consisting of $m+2$ terms. But, by the preceding principle, the common difference of this series will be equal to the difference of the extremes divided by the number of terms less one; that is, $d=\frac{b-a}{m+2-1}=\frac{b-a}{m+1}$; therefore, *the common difference will be equal to the difference of the two numbers, divided by the number of means plus one.*

Let it be required to insert five arithmetical means between 3 and 15.

Here $d=\frac{15-3}{5+1}=2$; hence the series is 3, 5, 7, 9, 11, 13, 15.

It is evident, that if we insert the same number of means between the consecutive terms of an arithmetical series, the result will form a new progression. Thus, if we insert 3 terms between the consecutive terms of the progression, 1, 9, 17, &c., the new series will be **1**, 3, 5, 7, **9**, 11, 13, 15, **17**, and so on.

EXAMPLES.

1. Find the sum of the natural series of numbers 1, 2, 3, 4, . . carried to 1000 terms. Ans. 500500.

2. Required the last term, and the sum of the series of odd numbers 1, 3, 5, 7, . . . continued to 101 terms. Ans. 201 and 10201.

3. How many times does a common clock strike, in a week? Ans. 1092.

4. Find the nth term, and the sum of n terms of the natural series of numbers 1, 2, 3, 4 . . . . Ans. n, and $\frac{1}{2}n(n+1)$.

5. Find the nth term, and the sum of n terms, of the series of odd numbers 1, 3, 5, 7. Ans. $2n-1$, and n^2.

6. The first and last terms of an arithmetical series are 2 and 29, and the common difference is 3; required the number of terms and the sum of the series. Ans. 10 and 155

7. The first and last terms of a decreasing arithmetical series are 10 and 6, and the number of terms 9; required the common difference, and the sum of the series. Ans. $\frac{1}{2}$ and 72.

8. The first term of a decreasing arithmetical series is 10, the number of terms 10, and the sum of the series 85; required the last term and the common difference. Ans. 7 and $\frac{1}{3}$.

9. Required the series obtained from inserting four arithmetical means between each of the two terms of the series 1, 16, 31, &c.
Ans. 1, 4, 7, 10, 13, 16, &c.

10. The sum of an arithmetical progression is 72, the first term is 24, and the common difference is —4; required the number of terms. Ans. 9 or 4.

In finding the value of n in this question, it is required to solve the equation $n^2-13n=-36$, which has two roots, 9 and 4. These give rise to the two following series, in both of which the sum is 72.

First series, 24, 20, 16, 12, 8, 4, 0, —4, —8.

Second series, 24, 20, 16, 12.

11. A man bought a farm, paying for the first acre 1 dollar, for the second 2 dollars, for the third 3 dollars, and so on; when he came to settle, he had to pay 12880 dollars; how many acres did the farm contain, and what was the average price per acre?
Ans. 160 acres, at $\$80\frac{1}{2}$ per acre.

12. If a person, A, start from a certain place, traveling a miles the first day, $2a$ the second, $3a$ the third, and so on; and at the end of 4 days, B start after him from the same place, traveling uniformly $9a$ miles a day; when will B overtake A?

Let $x=$ the number of days required; then the distance traveled by A in x days $=a+2a+3a$, &c., to x terms, $=\frac{1}{2}ax(x+1)$; and the distance traveled by B in $(x-4)$ days $=9a(x-4)$.
Whence $\frac{1}{2}ax(x+1)=9a(x-4)$. From which $x=8$, or 9.

Hence, B overtakes A at the end of 8 days; and since, on the ninth day, A travels $9a$ miles, which is B's uniform rate, they will be together at the end of the ninth day. This is an instance of the precision with which the solution of an equation points out the circumstances of a problem.

13. A sets out 3 hours and 20 minutes before B, and travels at the rate of 6 miles an hour; in how many hours will B overtake A, if he travel 5 miles the first hour, 6 the second, 7 the third, and so on? Ans. 8 hours.

14. Two travelers, A and B, set out from the same place, at the same time. A travels at the constant rate of 3 miles an hour, but B's rate of traveling, is 4 miles the first hour, $3\frac{1}{2}$ the second, 3 the third, and so on, in the same series; in how many hours will A overtake B? Ans. 5 hours.

REVIEW.—225. How do you insert m arithmetical means between two given numbers?

GEOMETRICAL PROGRESSION.

ART. **226.**—A Geometrical Progression is a series of terms, each of which is derived from the preceding, by multiplying it by a constant quantity, termed the *ratio*.

Thus, 1, 2, 4, 8, 16, &c., is an *increasing* geometrical series, whose common ratio is 2.

Also, 54, 18, 6, 2, &c., is a *decreasing* geometrical series, whose common ratio is $\frac{1}{3}$.

Generally, a, ar, ar^2, ar^3, &c., is a geometrical progression, whose common ratio is r, and which is an *increasing* or *decreasing* series, according as r is *greater*, or *less* than 1.

It is obvious, that the common ratio in any series, will be ascertained by dividing any term of the series, by that which immediately precedes it.

REMARK.—A geometrical progression is termed, by some writers, an *equirational series*, or a series of *continued proportionals*, or a *progression by quotients*.

ART. **227.**—To find the last term of the series.

Let a denote the first term, r the common ratio, l the nth term, and s the sum of n terms; then, the respective terms of the series will be

$$1,\ 2,\ 3,\ 4,\ 5 \ldots\ldots n-3,\ n-2,\ n-1,\quad n$$
$$a,\ ar,\ ar^2,\ ar^3,\ ar^4 \ldots\ldots ar^{n-4},\ ar^{n-3},\ ar^{n-2},\ ar^{n-1}.$$

That is, the exponent of r in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on; hence, the nth term of the series will be, $l=ar^{n-1}$; that is,

Any term of a geometric series is equal to the product of the first term, by the ratio raised to a power, whose exponent is one less than the number of terms.

EXAMPLES.

1. Find the 5th term of the geometric progression, whose first term is 4, and common ratio 3.

$3^4=3\times3\times3\times3=81$, and $81\times4=324$, the fifth term.

2. Find the 6th term of the progression 2, 8, 32, &c.

Ans. 2048.

3. Given the 1st term 1, and ratio 2, to find the 7th term.

Ans. 64.

4. Given the 1st term 4, and ratio 3, to find the 10th term.

Ans. 78732.

REVIEW.—226. What is a Geometrical Progression? Give examples of an increasing, and of a decreasing geometrical series. How may the common ratio in any geometrical series be found? 227. How is any term of a geometrical series found? Explain the principle of this rule.

5. Find the 9th term of the series, 2, 10, 50, &c. A. 781250.

6. Given the 1st term 8, and ratio $\frac{1}{2}$, to find the 15th term.

Ans. $\frac{1}{2048}$.

7. A man purchased 9 horses, agreeing to pay for the whole what the last would cost, at 2 dollars for the first, 6 for the second, &c.; what was the average price of each? Ans. $1458.

ART. **228.**—To find the sum of all the terms of the series.

If we multiply any geometrical series by the ratio, the result will be a new series, of which every term except the last, will have a corresponding term in the first series.

Thus, let a, ar, ar^2, ar^3, &c., be any geometrical series, and s its sum, then $s=a+ar+ar^2+ar^3 \ldots\ldots +ar^{n-2}+ar^{n-1}$

Multiplying this equation by r, we have

$$rs=ar+ar^2+ar^3+ar^4 \ldots\ldots +ar^{n-1}+ar^n.$$

The terms of the two series are identical, except the *first* term of the first series, and the *last* term of the second series. If, then, we subtract the first equation from the second, all the remaining terms of the series will disappear, and we shall have

$$rs-s=ar^n-a$$

Or $$(r-1)s=a(r^n-1)$$

Hence, $$s=\frac{a(r^n-1)}{r-1}$$

Since $l=ar^{n-1}$, we have $$rl=ar^n$$

Therefore, $$s=\frac{ar^n-a}{r-1}=\frac{rl-a}{r-1}.$$

Hence, the

RULE,

FOR FINDING THE SUM OF A GEOMETRICAL SERIES.

Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.

EXAMPLES.

1. Find the sum of 10 terms of the progression 2, 6, 18, 54, &c.
The last term $=2\times3^9=2\times19683=39366$.

$$s=\frac{lr-a}{r-1}=\frac{118098-2}{3-1}=59048. \text{ Ans.}$$

2. Find the sum of 7 terms of the progression 1, 2, 4, &c.

Ans. 127.

3. Find the sum of 10 terms of the progression 4, 12, 36, &c.

Ans. 118096.

4. Find the sum of 9 terms of the series 5, 20, 80, &c.

Ans. 436905.

5. Find the sum of 8 terms of the series, whose first term is $6\frac{1}{4}$, and ratio $\frac{3}{2}$. Ans. $307\frac{441}{512}$.

6. Find the sum of $8+20+50+$, &c., to 7 terms. A. $3249\frac{7}{8}$.
7. Find the sum of $3+4\frac{1}{2}+6\frac{3}{4}+$, &c., to 5 terms. A. $39\frac{9}{16}$.

REMARK.—If the ratio r is less than 1, the progression is decreasing, and the last term lr is less than a. In order that both terms of the fraction $\frac{rl-a}{r-1}$ shall be positive, the signs of the terms must be changed, and we have $s=\frac{a-rl}{1-r}$. The sum of the series when the progression is decreasing, is, therefore, found by the same rule, as when it is increasing, except that the product of the last term by the ratio, is to be subtracted from the first term, and the ratio subtracted from unity, instead of subtracting unity from the ratio.

8. Find the sum of 15 terms of the series 8, 4, 2, 1, &c. Ans. $15\frac{2047}{2048}$.
9. Find the sum of 6 terms of the series 6, $4\frac{1}{2}$, $3\frac{3}{8}$, &c. Ans. $19\frac{373}{512}$.

ART. **229.**—The formula $s=\frac{a-ar^n}{1-r}$, by separating the numerator into two parts, may be placed under the form

$$s=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now, when r is less than 1, it must be a proper fraction, which may be represented by $\frac{p}{q}$; then $r^n=\left(\frac{p}{q}\right)^n=\frac{p^n}{q^n}$. Since p is less than q, the higher the power to which the fraction is raised, the less will be the numerator compared with the denominator; that is, the less will be the value of the fraction; therefore, when n becomes *very large*, the value of $\frac{p^n}{q^n}$, or r^n will be *very small;* and, when n becomes *infinitely great*, the value of $\frac{p^n}{q^n}$, or r^n, will be *infinitely small*, that is, 0. But, when the numerator of a fraction is zero, its value is 0. This reduces the value of s, to $\frac{a}{1-r}$. Hence, *when the number of terms of a decreasing geometrical series is infinite, the last term is zero, and the sum is equal to the first term divided by one minus the ratio.*

REVIEW.—228. What is the rule for finding the sum of the terms of a geometrical series? Explain the reason of this rule. When the series is decreasing, how must the formula, expressing the sum, be written, so that both terms of the fraction may be positive? 229. What is the rule for finding the sum of a decreasing geometrical series, when the number of terms is infinite? Explain the reason of this rule.

1. Find the sum of the infinite series $1+\frac{1}{3}+\frac{1}{9}+$, &c.

Here $a=1$, $r=\frac{1}{3}$, and $s=\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$. Ans.

2. Find the sum of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$, &c. Ans. 2.

3. Find the sum of the infinite series $9+6+4+$, &c. A. 27.

4. Find the sum of the infinite series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+$, &c. Ans. $\frac{3}{4}$.

5. Find the sum of the infinite series $1+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+$, &c.

Ans. $\frac{x^2}{x^2-1}$.

6. Find the sum of the infinite geometrical progression $a-b+\frac{b^2}{a}-\frac{b^3}{a^2}+\frac{b^4}{a^3}-$, &c., in which the ratio is $-\frac{b}{a}$. Ans. $\frac{a^2}{a+b}$.

7. If a body moves 10 feet the first second, 5 the next second, $2\frac{1}{2}$ the next, and so on, continually, how many feet would it move over? Ans. 20.

ART. **230.**—The two equations, $l=ar^{n-1}$, and $s=\frac{ar^n-a}{r-1}$, furnish this general problem: *knowing three of the five quantities a, r, n, l, and s, of a geometrical progression, to determine the other two.* This problem embraces ten different questions, as in arithmetical progression. Some of the cases, however, involve the extraction of high roots, the application of logarithms, and the solution of higher equations than have been treated of in the preceding pages. The following is one of the most simple and useful of these cases.

Having given the first and last terms, and the number of terms of a geometrical progression, to find the ratio.

Here $$l=ar^{n-1}, \text{ or } r^{n-1}=\frac{l}{a}$$

Hence, $$r=\sqrt[n-1]{\left(\frac{l}{a}\right)}.$$

1. The first and last terms of a geometrical series, are 3 and 48, and the number of terms 5; required the intermediate terms.

Here $$l=48,\ a=3,\ n-1=5-1=4$$

Hence, $r^4=\frac{48}{3}=16$, and $r^2=\sqrt{16}=4$, and $r=\sqrt{4}=2$.

2. In a geometrical series of three terms, the first and last terms are 4 and 16; required the middle term. Ans. 8.

In a geometrical progression, containing three terms, the middle term is called a mean proportional between the other two.

3. Find a mean proportional between 8 and 32. Ans. 16.

4. The first and last terms of a geometrical series are 2 and 162, and the number of terms 5; required the ratio. Ans. 3

RATIO AND PROPORTION.

ART. **231.**—Two quantities of the same kind, may be compared in two ways:

1st. By finding *how much* the one *exceeds* the other.

2d. By finding *how many times* the one *contains* the other.

If we compare the numbers 2 and 6, by the first method, we say that 2 is 4 *less* than 6, or that 6 is 4 *greater* than 2.

If we compare 2 and 6, by the second method, we say that 6 is equal to *three times* 2, or that 2 is *one third* of 6. This method of comparison gives rise to proportion.

ART. **232.**—*Ratio* is the quotient which arises from dividing one quantity by another of the *same kind.* Thus, the ratio of 2 to 6 is 3; the ratio of a to ma is m.

REMARKS.—1st. In comparing two numbers or quantities by their quotient, the number expressing the ratio which the first bears to the second, will depend on which is made the *standard* of comparison. Thus, in comparing 2 and 6, if we make 2 the *unit* of measure, or standard, we find, that 6 is *three times* the standard. If we make 6 the unit of measure, or standard, we find, that 2 is *one third* of the standard. In finding the ratio of one number to another, the French mathematicians always make the *first* of the two numbers the standard of comparison; while the English make the last named the standard. Thus, the French say the ratio of 2 to 6 is 3; while the English say it is $\frac{1}{3}$. The French method is now generally used in the United States, though, in a few works, the other is still retained.

2d. In order that two quantities may be compared, or have a ratio to each other, it is evidently necessary that they should be of the same kind, so that one may be some part of, or some number of times the other. Thus, 2 *yards* has a ratio to 6 *yards,* because the latter is *three times* the former; but 2 *yards* has no ratio to 6 *dollars,* since the one can not be said to be either greater, less, or any number of times the other.

ART. **233.**—When two numbers, as 2 and 6, are compared, the *first* is called the *antecedent,* and the *second* the *consequent.*

An antecedent and consequent, when spoken of as *one,* are called a *couplet.* When spoken of as *two,* they are called the *terms* of the ratio. Thus, 2 and 6 together, form a couplet, of which 2 is the first term, and 6 the second.

ART. **234.**—Ratio is expressed in two ways.

1st. In the form of a fraction, of which the *antecedent* is the *denominator,* and the *consequent* the *numerator.* Thus, the ratio of 2 to 6, is expressed by $\frac{6}{2}$; the ratio of 3 to 12, by $\frac{12}{3}$, &c.

REVIEW.—231. In how many ways, may two quantities of the same kind be compared? Compare the numbers 2 and 6 by the first method. By the second. 232. What is ratio? Give an illustration. 233. When two numbers are compared, what is the first called? The second? Give an example.

2d. By placing two points (:) between the terms of the ratio. Thus, the ratio of 2 to 6, is written 2 : 6; the ratio of 3 to 8, 3 : 8, &c.

ART. **235.**—The ratio of two quantities, may be either a whole number, a common fraction, or an *interminate* decimal.

Thus, the ratio of 2 to 6 is $\frac{6}{2}$, or 3.

The ratio of 10 to 4 is $\frac{4}{10}$, or $\frac{2}{5}$.

The ratio of 2 to $\sqrt{5}$ is $\frac{\sqrt{5}}{2}$, or $\frac{2.236+}{2}$, or 1.118+.

We see, from this, that the ratio of two quantities can not always be expressed exactly, except by symbols; but, by taking a sufficient number of decimal places, it may be found to any required degree of exactness.

ART. **236.**—Since the ratio of two numbers is expressed by a fraction, of which the antecedent is the denominator, and the consequent the numerator, it follows, that whatever is true with regard to a fraction, is true with regard to the terms of a ratio. Hence,

1st. *To multiply the consequent, or to divide the antecedent of a ratio by any number, multiplies the ratio by that number.* (Articles 122, 125.)

Thus, the ratio of 4 to 12, is 3.

The ratio of 4 to 12×5, is 3×5.

The ratio of 4÷2 to 12, is 6, which is equal to 3×2.

2d. *To divide the consequent, or to multiply the antecedent of a ratio by any number, divides the ratio by that number.* (Articles 123, 124.)

Thus, the ratio of 3 to 24, is 8.

The ratio of 3 to 24÷2, is 4, which is equal to 8÷2.

The ratio of 3×2 to 24, is 4, which is equal to 8÷2.

3d. *To multiply, or divide, both the antecedent and consequent of a ratio by any number, does not alter the ratio.* (Articles 126, 127.)

Thus, the ratio of 6 to 18, is 3.

The ratio of 6×2 to 18×2, is 3.

The ratio of 6÷2 to 18÷2, is 3.

ART. **237.**—When the two numbers are *equal*, the ratio is said to be a ratio of *equality*. When the second number is greater than

REVIEW.—234. When are the antecedent and consequent of a ratio called a couplet? When the terms of a ratio? By what two methods is ratio expressed? Give an example. 235. What forms may the ratio of two quantities have? 236. How is a ratio affected by multiplying the consequent, or dividing the antecedent? Why? How is a ratio affected by dividing the consequent, or multiplying the antecedent? Why? How is a ratio affected, by either multiplying or dividing both antecedent and consequent by the same number? Why?

the first, the ratio is said to be a ratio of *greater inequality*, and when it is less, the ratio is said to be a ratio of *less inequality.*

Thus, the ratio of 4 to 4, is a ratio of equality.

The ratio of 4 to 8, is a ratio of greater inequality.

The ratio of 4 to 2, is a ratio of less inequality.

We see, from this, that a ratio of equality may be expressed by 1; a ratio of greater inequality, by a number greater than 1; and a ratio of less inequality, by a number less than 1.

ART. **238.**—When the corresponding terms of two or more ratios are multiplied together, the ratios are said to be *compounded*, and the result is termed a *compound ratio.* Thus, the ratio $\frac{10}{3}$, compounded with the ratio $\frac{6}{5}$, is $\frac{10}{3}\times\frac{6}{5}=\frac{60}{15}=4$. In this case, 3 multiplied by 5, is said to have to 10 multiplied by 6, the ratio compounded of the ratios of 3 to 10 and 5 to 6.

ART. **239.**—Ratios may be compared with each other, by reducing the fractions which represent them, to a common denominator. Thus, to ascertain whether the ratio of 2 to 5 is greater than the ratio of 3 to 8, we have the two fractions, $\frac{5}{2}$ and $\frac{8}{3}$, which being reduced to a common denominator, are $\frac{15}{6}$ and $\frac{16}{6}$; and, since the first is less than the second, we infer, that the ratio of 2 to 5 is less than the ratio of 3 to 8.

PROPORTION.

ART. **240.**—Proportion is an equality of ratios. Thus, if a, b, c, d are four quantities, such that $\frac{b}{a}$ is equal to $\frac{d}{c}$, then a, b, c, d form a proportion, and we say that a is to b, as c is to d; or, that a has the same ratio to b, that c has to d.

Proportion is written in two ways.

1st. By placing the double colon between the ratios. Thus,

$$a : b : : c : d.$$

2d. By placing the sign of equality between them. Thus,

$$a : b = c : d.$$

The first method is the one generally used.

From the preceding definition, it follows, that when four quantities are in proportion, the second divided by the first, gives the same quotient as the fourth divided by the third. This is the *test* of the proportionality of four quantities. Thus, if 3, 6, 5, 10 are

REVIEW.—237. What is a ratio of equality? Of greater inequality? Of less inequality? Give examples. 238. When are two or more ratios said to be compounded? Give an example. 239. How may ratios be compared to each other? 240. What is proportion? Give an example. How are four quantities in proportion written? Give examples.

the four terms of a true proportion, so that $3 : 6 :: 5 : 10$, we must have $\frac{6}{3}=\frac{10}{5}$.

If these fractions are equal to each other, the proportion is *true*, if they are not equal to each other, it is *false*.

Thus, let it be required to find whether $3 : 8 :: 2 : 5$.

The first ratio is $\frac{8}{3}$, the second is $\frac{5}{2}$, or $\frac{16}{6}$, and $\frac{15}{6}$; therefore, 3, 8, 2, 5 are not proportional quantities.

REMARK.—The words *ratio* and *proportion*, in common language, are sometimes confounded with each other. Thus, two quantities are said to be in the *proportion* of 3 to 4, instead of, in the *ratio* of 3 to 4. A ratio subsists between *two* quantities, a proportion only between *four*. It requires *two equal ratios* to form a proportion.

ART. **241.**—In the proportion $a : b :: c : d$, each of the quantities a, b, c, d, is called a *term*. The first and last terms are called the *extremes*, the second and third, the *means*.

ART. **242.**—Of four proportional quantities, the first and third are called *antecedents*, and the second and fourth, *consequents* (Art. 233); and the last is said to be a fourth proportional to the other three, taken in their order.

ART. **243.**—Three quantities are in proportion, when the first has the same ratio to the second, that the second has to the third. In this case, the *middle* term is called a *mean proportional* between the other two. Thus, if we have the proportion

$$a : b :: b : c$$

then b is called a *mean proportional* between a and c, and c is called a *third proportional* to a and b.

ART. **244.**—PROPOSITION I.—*In every proportion, the product of the means is equal to the product of the extremes.*

Let $a : b :: c : d$.

Then, since this is a true proportion, the quotient of the second divided by the first, is equal to the quotient of the fourth divided by the third. Therefore, we must have

$$\frac{b}{a}=\frac{d}{c}.$$

Multiplying both sides of this equality by ac, to clear it of fractions, we have $\frac{abc}{a}=\frac{adc}{c}$. Or, $bc=ad$.

Illustration by numbers. $3 : 6 :: 5 : 10$, and $6\times5=3\times10$.

REVIEW.—240. Give examples of a true and false proportion. What is a test of the proportionality of four quantities? 241. What are the first and last terms of a proportion called? The second and third terms? 242. What are the first and third terms of a proportion called? The second and fourth? 243. When are three quantities in proportion? Give an example. What is the second term called? The third?

From the equation $bc=ad$, we have $d=\frac{bc}{a}$, $c=\frac{ad}{b}$, $b=\frac{ad}{c}$, and $a=\frac{bc}{d}$, from which we see, that if any three terms of a proportion are given, the fourth may be readily found.

The first three terms of a proportion, are ac, bd, and $acxy$; what is the fourth? Ans. $bdxy$.

REMARK.—This proposition furnishes a more convenient *test* of the proportionality of four quantities, than the method given in Article 240. Thus, to ascertain whether $3:8::2:5$, it is merely necessary to compare the product of the means and the extremes; and, since 3×5 is not equal to 8×2, we infer that the proportion is *false*.

ART. **245.**—PROPOSITION II.—Conversely, *If the product of two quantities is equal to the product of two others, two of them may be made the means, and the other two the extremes of a proportion.*

Let $bc=ad$.

Dividing each of these equals by ac, we have

$$\frac{bc}{ac}=\frac{ad}{ac}; \text{ Or, } \frac{b}{a}=\frac{d}{c}.$$

That is, $a:b::c:d$.

Illustration. $5\times8=4\times10$, and $4:5::8:10$.

ART. **246.**—PROPOSITION III.—*If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.*

If $a:b::b:c$

Then, by Art. 244, $ac=bb=b^2$.

It follows, from Art. 245, that the converse of this proposition is also true. Thus, if $ac=b^2$,

Then, $a:b::b:c$.

That is, *if the product of the first and third of two quantities, is equal to the square of a second, the first is to the second, as the second is to the third.*

Illustration. If $4:6::6:9$, then $4\times9=6^2=36$.

If $2\times8=16$, then $2:\sqrt{16}::\sqrt{16}:8$

Or $2:4::4:8$.

ART. **247.**—PROPOSITION IV.—*If four quantities are in proportion, they will be in proportion by* ALTERNATION; *that is, the first will have the same ratio to the third, that the second has to the fourth.*

Let $a:b::c:d$.

This gives, $\frac{b}{a}=\frac{d}{c}$.

Multiplying both sides by c, $\frac{bc}{a}=d$.

Dividing both sides by b, $\frac{c}{a}=\frac{d}{b}$.

That is, $a:c::b:d$.

Illustration. $2:7::6:21$, and $2:6::7:21$.

ART. **248.**—PROPOSITION V.—*If four quantities are in proportion, they will be in proportion by* INVERSION; *that is, the second will be to the first as the fourth to the third.*

Let $a:b::c:d$.

By Art. 244, $ad=bc$.

Dividing both sides by b, $\frac{ad}{b}=c$.

Dividing both sides by d, $\frac{a}{b}=\frac{c}{d}$.

That is, $b:a::d:c$.

Illustration. $2:5::6:15$, and $5:2::15:6$.

ART. **249.**—PROPOSITION VI.—*If two sets of proportions have an antecedent and consequent in the one, equal to an antecedent and consequent in the other, the remaining terms will be proportional.*

Let $a:b::c:d$ (1.)

And $a:b::e:f$ (2.)

Then will $c:d::e:f$

For, from 1st proportion $\frac{b}{a}=\frac{d}{c}$;

From 2d proportion, $\frac{b}{a}=\frac{f}{e}$.

Hence, $\frac{d}{c}=\frac{f}{e}$.

This gives, $c:d::e:f$

Illustration. $3:5::6:10$

$3:5::9:15$

And $6:10::9:15$.

REMARK.—This proposition is generally termed *equality of ratios.* It is almost *self-evident.*

ART. **250.**—PROPOSITION VII.—*If four quantities are in proportion, they will be in proportion by* COMPOSITION; *that is, the sum of the first and second, will be to the second, as the sum of the third and fourth, is to the fourth.*

Let $a:b::c:d$

Then will $a+b:b::c+d:d$

From the 1st proportion, $bc=ad$, by Art. 244.

Adding bd to each, $bd=bd$,

$bc+bd=ad+bd$; Or $b(c+d)=d(a+b)$.

Dividing each side by $c+d$, $b=\frac{d(a+b)}{c+d}$;

By $a+b$, $\frac{b}{a+b}=\frac{d}{c+d}$.

This gives, $a+b : b :: c+d : d$.

Illustration. $3 : 4 :: 6 : 8$

$3+4 : 4 :: 6+8 : 8$; Or, $7 : 4 :: 14 : 8$.

REMARK.—In a similar manner, it may be proved, that the sum of the first and second terms, will be to the *first*, as the sum of the third and fourth is to the *third*.

ART. 251.—PROPOSITION VIII.—*If four quantities are in proportion, they will be in proportion by* DIVISION; *that is, the difference of the first and second, will be to the second, as the difference of the third and fourth is to the fourth.*

Let $a : b :: c : d$,

Then will $a-b : b :: c-d : d$.

From the 1st proportion, $bc=ad$, by Art. 244.

Subtracting bd from each, $bd=bd$

$bc-bd=ad-bd$;

Or, $b(c-d)=d(a-b)$.

Dividing each side by $c-d$, $b=\frac{d(a-b)}{c-d}$;

By $a-b$ $\frac{b}{a-b}=\frac{d}{c-d}$.

This gives, $a-b : b :: c-d : d$.

Illustration. $8 : 5 :: 16 : 10$

$8-5 : 5 :: 16-10 : 10$; Or, $3 : 5 :: 6 : 10$.

REMARK.—In a similar manner, it may be proved, that the difference of the first and second will be to the *first*, as the difference of the third and fourth is to the *third*.

ART. 252.—PROPOSITION IX.—*If four quantities are in proportion, the sum of the first and second will be to their difference, as the sum of the third and fourth is to their difference.*

Let $a : b :: c : d$, (1.)

Then will $a+b : a-b :: c+d : c-d$.

From the 1st, by composition, Art. 250,

$a+b : b :: c+d : d$.

By alternation, $a+b : c+d :: b : d$, Art. 247.

This gives, $\dfrac{c+d}{a+b}=\dfrac{d}{b}$.

From the 1st, by division,

$$a-b : b :: c-d : d,$$

By alternation, $a-b : c-d :: b : d$;

This gives, $\dfrac{c-d}{a-b}=\dfrac{d}{b}$; hence $\dfrac{c+d}{a+b}=\dfrac{c-d}{a-b}$.

That is, $a+b : c+d :: a-b : c-d$;

Or, by alternation, $a+b : a-b :: c+d : c-d$.

Illustration. $5 : 3 :: 10 : 6$

$$5+3 : 5-3 :: 10+6 : 10-6$$

Or, $8 : 2 :: 16 : 4$.

ART. **253.**—PROPOSITION X.—*If four quantities are in proportion like powers, or roots, of those quantities, will also be in proportion.*

Let $a : b :: c : d$.

Then will $a^n : b^n :: c^n : d^n$

For, since $\dfrac{b}{a}=\dfrac{d}{c}$.

If we raise each of these equals to the nth power, we have,

$$\frac{b^n}{a^n}=\frac{d^n}{c^n}.$$

That is, $a^n : b^n :: c^n : d^n$,

Where n may either be a whole number or a fraction.

Illustration. $2 : 3 :: 4 : 6$

$$2^2 : 3^2 :: 4^2 : 6^2$$

Or $4 : 9 :: 16 : 36$

Also, $a^2 : b^2 :: m^2a^2 : m^2b^2$

And $\sqrt{a^2} : \sqrt{b^2} :: \sqrt{m^2a^2} : \sqrt{m^2b^2}$

Or $a : b :: ma : mb$.

ART. **254.**—PROPOSITION XI.—*If two sets of quantities are in proportion, the products of the corresponding terms will also be in proportion.*

Let $a : b :: c : d$, (1.)

And $m : n :: r : s$; (2.)

Then will $am : bn :: cr : ds$.

For, from the 1st, $\dfrac{b}{a}=\dfrac{d}{c}$; and from the 2d, $\dfrac{n}{m}=\dfrac{s}{r}$.

Multiplying these equals together,

$$\frac{b}{a}\times\frac{n}{m}=\frac{d}{c}\times\frac{s}{r}, \text{ or } \frac{bn}{am}=\frac{ds}{cr}.$$

This gives, $am : bn :: cr : ds$.

Illustration. $3 : 5 :: 6 : 10,$

$4 : 3 :: 8 : 6,$

$12 : 15 :: 48 : 60.$

ART. **255.**—PROPOSITION XII.—*In any continued proportion, that is, any number of proportions having the same ratio, any one antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b :: c : d :: m : n,$ &c.

Then will $a : b :: a+c+m : b+d+n;$

Since $a : b :: c : d$, we have $bc=ad$.

Since $a : b :: m : n$, we have $bm=an$.

Adding ab to each, $ab=ab$. The sums of these equalities give $ab+bc+bm=ab+ad+an;$

Or $b(a+c+m)=a(b+d+n)$.

Dividing by $a+c+m$, $b=\dfrac{a(b+d+n)}{a+c+m};$

Dividing both sides by a $\dfrac{b}{a}=\dfrac{b+d+n}{a+c+m}.$

This gives, $a : b :: a+c+m : b+d+n.$

Illustration. $3 : 4 :: 6 : 8 :: 9 : 12$

$3 : 4 :: 3+6+9 : 4+8+12$

Or $3 : 4 :: 18 : 24.$

REMARK.—In the preceding demonstrations, the proof has generally been made to involve the definition of proportion, that is, that the four quantities, a, b, c, d, are in proportion, when $\frac{b}{a}=\frac{d}{c}$. This is regarded as a matter of great importance to the pupil. If the instructor chooses to dispense with this, as some writers do, several of the demonstrations may be somewhat shortened. There are several other Propositions in Proportion, that may be easily demonstrated, in a manner similar to the preceding, but they are of so little use, as not to be worthy of the pupil's attention.

THE END.